

Recursive towers of curves over finite fields using graph theory

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Abstract

We give a new way to study recursive towers of curves over a finite field, defined from a bottom curve X and a correspondence Γ on X . In particular, we study their asymptotic behavior. A close examination of singularities leads to a necessary condition for a tower to be asymptotically good. Then, spectral theory on a directed graph and considerations on the class of Γ in $\text{NS}(X \times X)$ lead to the fact that, under some mild assumptions, a recursive tower which does not reach Drinfeld-Vladut bound cannot be optimal in Tsfasmann-Vladut sense. Results are applied to the Bezerra-Garcia-Stichtenoth tower along the paper for illustration.

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Introduction

Since Garcia-Stichtenoth's well-known Inventiones'95 paper appeared [GS95], many other examples of *good* recursive towers of curves over finite fields were described in the literature. Recall that a tower \mathcal{T} of smooth projective absolutely irreducible curves over a finite field is said to be *good* if it has many rational points over some finite extension of the base field, that is if there exists $r \geq 1$ such that $\lambda_r(\mathcal{T}) > 0$ (cf. §1 for precise definitions). More precisely, the closer to zero is the deficiency (see equation (4) below), the better is the tower. Towers reaching the Drinfeld-Vladut bound over some finite extension of the base field have deficiency zero, hence are optimal. Some recursive towers reach Drinfeld-Vladut bound for q square, for instance in [GS95, GSR03, Gar96], others give interesting non-zero lower bounds for $A(q)$ for some non-square values of q , for instance [BGS05, BS07, BGS08] for $A(q^3)$. It turns out that all these towers are recursive over the projective line \mathbb{P}^1 : they are given by an explicit correspondence Γ in $\mathbb{P}^1 \times \mathbb{P}^1$, and the curves of the tower are — sometimes irreducible components of — the normalizations of the curves

$$C_n = \{(P_1, \dots, P_n) \in (\mathbb{P}^1)^n \mid (P_i, P_{i+1}) \in \Gamma, \text{ for each } i = 1, \dots, n-1\}.$$

The point is that no author give the procedure they used to obtain, or merely to guess which explicit equation will lead to a good recursive tower. For almost twenty years, very few papers

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contain theoretical considerations on recursive towers. The first small family of exceptions are a series of papers from Elkies [Elk01, LMSE02], whose goal is to make it possible that any good recursive tower should come from the modular world. Another very small family of exceptions are Lenstra's [Len02] and the subsequent Beelen's [Bee04] papers, who deal with possibilities of getting recursive towers with a great number of rational points. The last exception is Bouw and Beelen's paper [BB05], who give a link between some good recursive towers and Picard-Fuchs differential equations in characteristic p . Up to our knowledge, these are the only theoretical studies of recursive towers. The reader is referred to the excellent survey of Li [Li10] for details.

Section 1 is only a set-up one. We fix notations, introduce the standard invariants of a tower and state some common hypothesis for most statements. The definition of a recursive tower requires only a pair (X, Γ) where X is a smooth, projective, absolutely irreducible curve defined over \mathbb{F}_q and where Γ is a correspondence on $X \times X$ which is supposed to be absolutely irreducible, reduced, and of type (d, d) for $d \geq 2$ (see §1.1, for precise definition). Up to now in the literature the curve X is the projective line \mathbb{P}^1 but there is no reason for restricting ourselves to this case.

The aim of the remaining sections is firstly to understand better which features of the data (X, Γ) can lead to a good recursive tower, then to study up to which point a recursive tower can be good. The key ingredients are considerations on singular models of the tower, the use of spectral theory on directed graphs and the geometry of the surface $X \times X$ through the class of Γ in the Neron-Severi group $\text{NS}(X \times X)$.

In section 2, we focus on the geometry of a recursive tower. Most previous authors chose the function field point of view. In doing so, a large part of the geometry of the tower disappears. We investigate more closely this geometry. This leads us to distinguish three models of towers: the *singular* one, the *smooth* one, and finally the *sharp* one which is an avatar of the singular model. Of course, the smooth model — which corresponds to the usual tower of function fields — is the most interesting. At any stage, the three curves are birational. The sharp one being naturally embedded in a smooth surface, we can evaluate by adjunction formula and desingularization the geometric genus sequence of the tower. We deduce some necessary conditions for a tower to be good:

Proposition. *Let (X, Γ) be a correspondence as in section 1.1 and let $\mathcal{T} = (C_n)_{n \geq 1}$ be the associated recursive tower. Suppose that C_n is irreducible for any $n \geq 1$ and that the sequence genus $(g_n)_{n \geq 1}$ tends to infinity. If there is at least one $r \geq 1$ such that $\lambda_r(\mathcal{T}) > 0$, then*

- (i) *either C_n is singular for any n greater than some n_0 ;*
- (ii) *either $g_1 = g(X) \geq 2$ and both covers $\pi_i : \Gamma \rightarrow X$ for $i = 1, 2$ are étale over X .*

More precisely, in the singular case, we evaluate how the *global measure of singularity* should grow when $n \rightarrow \infty$, for a tower to be good. A key point for the following of the paper is the understanding of the singular points. We characterize them, and we study other singular points on the intersection of the tower curves with an hypersurfaces for later use.

In section 3, we associate to each recursive tower an infinite directed graph. It depends only on the base curve X and on the correspondence Γ . This graph is closely related, but different, to the one introduced by Beelen [Bee04]. The main difference is that the former depends on the singular model of the tower, while the later depends on the smooth model, that is on the associated function fields tower. Though we share some common observations with Beelen, we give some new applications of the graph. It is a very convenient way to represent a tower — in some way better than the equations themselves —, in the sense that most important properties can be directly seen from it. The degree, the singular points, the totally splitting points, sometimes the irreducibility, can directly be read off the graph. As an illustration of the

effectiveness of this point of view, we end this section by the study of the *BGS tower* over \mathbb{F}_{p^3} attaining the Zink's lower bound [BGS05, BS07, BGS08], with the help of its graph. This tower appears several times in this paper to illustrate our point of view. In this section, we compute a nice closed formula for its geometric genus sequence, using the adjunction method rather than the usual "Riemann-Hurwitz" one. Note that only an upper bound was given in the three previous papers devoted to this tower. This closed formula will be used in the last section to compute exactly some invariants of this tower defined by Tsfasman and Vladut in [TV02].

Section 4 is devoted to the application to the asymptotic behavior of a recursive tower. Aside from the cycle singularities study in section 2, we use spectral theory of non-negative matrices (also called Perron-Frobenius theory) and intersection theory on $X \times X$. We give the number of cycles counted with multiplicities of given length in the graph. On the other hand, under some assumption on the graph, Perron-Frobenius theory applied to adjacency matrices of some well chosen finite component of the graph leads us to a lower-bound of the number of cycles (counted without multiplicities). Altogether, these bounds allow us to prove an important property of the graph of a recursive tower:

Theorem. *Let (X, Γ) be a correspondence as in section 1.1 such that the curves C_n of the associated tower are all irreducible. Then the graph $\mathcal{G}_\infty(X, \Gamma)$ has at most one finite d -regular strongly connected component.*

Next we deduce from this result a specific property of the asymptotic behavior of a recursive tower under some assumption, whose meaning is that, unless the tower reaches Drinfeld-Vladut bound on some finite extension of \mathbb{F}_q , it will definitely be not optimal!

Theorem. *Let (X, Γ) be a correspondence as in section 1.1. Suppose that the curves C_n of the associated tower are all irreducible and that the geometric genus sequence $(g_n)_{n \geq 1}$ goes to $+\infty$. Suppose also that:*

- (i) *the singular points of C_n give rise to a number of geometric points in \tilde{C}_n negligible in front of d^n for large n .*
- (ii) *the graph \mathcal{G}_∞ contains at least one (hence exactly one by theorem 23) finite d -regular strongly connected component.*

Then, there exists at most one integer $r \geq 1$ such that $\beta_r \neq 0$.

Note that the hypotheses of this theorem are satisfied by a large part of the known recursive good towers. This is the case of the BGS tower (see loc. cit.). Combining with the closed formula for the geometric genus of this tower, we are able to compute exactly two invariants, its defect δ and its zeta function both defined in [TV02].

1 Models of recursive towers

1.1 The setting

Let X be a smooth projective absolutely irreducible algebraic curve of genus $g_X \geq 0$ defined over the finite field \mathbb{F}_q . Let Γ be a correspondence of type (d_1, d_2) on the surface $X \times X$; this means that $d_1 = \Gamma \cdot H$ and $d_2 = \Gamma \cdot V$, where $H = X \times \text{pt}$ and $V = \text{pt} \times X$ denote the horizontal and vertical divisors on $X \times X$ (cf. [Har77, Chap V, §1, Ex 1.9, page 368]). In the whole paper we make the following assumptions:

Hypotheses — The curve X is supposed to be smooth, projective, absolutely irreducible, and defined over \mathbb{F}_q . The correspondence Γ on $X \times X$ is supposed to be absolutely irreducible, reduced, and of type (d, d) for $d \geq 2$.

Consider the two projection morphisms $\pi_i : X \times X \rightarrow X$, defined by $\pi_i(P_1, P_2) = P_i$ for $i = 1, 2$. We have the following diagram:

$$\begin{array}{ccc}
 & X \times X & \\
 \pi_1 \swarrow & & \searrow \pi_2 \\
 X & & X
 \end{array}
 \quad \text{which restricts to } \Gamma \text{ has finite morphisms: }
 \begin{array}{ccc}
 & \Gamma & \\
 \pi_1 \swarrow & & \searrow \pi_2 \\
 X & & X
 \end{array}
 \quad \begin{array}{l} \deg(\pi_1) = d_2 \\ \deg(\pi_2) = d_1 \end{array}$$

Note that the curve Γ is not supposed to be smooth. The irreducibility assumption of Γ is natural since we will deal with irreducible towers. The assumption $d_1 = d_2 = d$ is a necessary condition in order to obtain good recursive towers, as will follow from lemma 2.

In most of the examples studied in the literature, the base curve X is the projective line \mathbb{P}^1 . As for the correspondence, it has most often separated variables, that is:

$$\Gamma_{f,g} = \{(P, Q) \in \mathbb{P}^1 \times \mathbb{P}^1 \mid f(P) = g(Q)\}$$

where f and g are two rational functions on \mathbb{P}^1 .

1.2 The smooth, the singular and the sharp models of a tower

From these datas, one can define three towers of curves. The *smooth* one is the most interesting. This is the tower studied by previous authors usually using the function field language, and will actually be also our target tower. However, a geometric description can be fruitful. For this purpose, we introduce the *singular* tower, naturally defined in geometric terms from the datas (X, Γ) by (1). This is in general a tower of singular curves. In order to study the genus of the smooth tower using adjunction formula on smooth surfaces and normalization process, it is useful to introduce also an intermediate *sharp* tower.

Let (X, Γ) be a correspondence as in section 1.1.

- The *singular-recursive tower* $\mathcal{T}(X, \Gamma)$ is the sequence of curves $(C_n)_{n \geq 1}$ defined by:

$$C_n = \{(P_1, P_2, \dots, P_n) \in X^n \mid (P_i, P_{i+1}) \in \Gamma \text{ for each } i = 1, 2, \dots, n-1\} \quad (1)$$

By definition, each curve C_n is embedded in the n -fold product X^n . For $1 \leq i \leq n$, let $\pi_i^n : C_n \rightarrow X$ (or simply π_i if the domain is clear from the context) be the i -th projection defined by $(P_1, \dots, P_n) \mapsto P_i$.

Thus $C_1 = X$ is supposed to be smooth, while $C_2 = \Gamma$ is not! So except for C_1 , the curves C_n for $n \geq 2$ need not to be smooth (even if Γ is, see propositions 6 and 11).

- The *smooth recursive tower* $\tilde{\mathcal{T}}(X, \Gamma)$ is the sequence of smooth curves $(\tilde{C}_n)_{n \geq 1}$ where, for each $n \geq 1$, we denote by \tilde{C}_n the normalization of the curve C_n and by $\nu_n : \tilde{C}_n \rightarrow C_n$ the desingularization morphism.

- The *sharp-recursive tower* $\mathcal{T}^\sharp(X, \Gamma)$ is the sequence of curves $(C_n^\sharp)_{n \geq 1}$, where C_n^\sharp is the pullback of the embedding $\Gamma \hookrightarrow X \times X$ along $\pi_{n-1}^{n-1} \circ \nu_n \times \text{Id} : \tilde{C}_{n-1} \times X \rightarrow X \times X$. It is also the

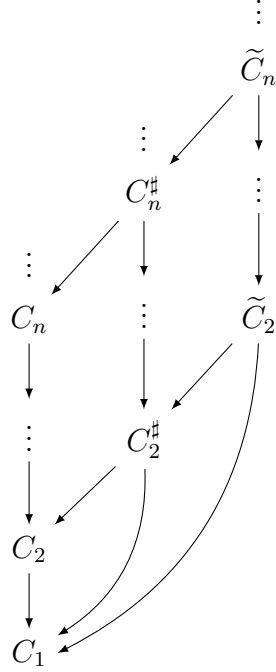


Figure 1: The three towers

pullback of the embedding $C_n \hookrightarrow C_{n-1} \times X$ along $\nu_n \times \text{Id} : \tilde{C}_n \times X \rightarrow C_n \times X$, so that we have the cartesian diagram

$$\begin{array}{ccc}
 C_n^\# & \hookrightarrow & \tilde{C}_{n-1} \times X \\
 \downarrow & & \downarrow \\
 C_n & \hookrightarrow & C_{n-1} \times X \\
 \downarrow & & \downarrow \\
 \Gamma & \hookrightarrow & X \times X
 \end{array}$$

Note that the curve $C_n^\#$ is singular, but less than C_n itself.

Since our final goal is to study the asymptotic behavior of smooth absolutely irreducible curves, we will assume in most statements that the singular curves C_n are irreducible. Up to our knowledge, the only important reducible recursive tower containing a good irreducible sub-tower is in [BGS05]. However, this good irreducible sub-tower turned later to be itself recursive in [BGS08].

A simple criterion asserting this irreducibility, satisfied by most known good recursive towers, is given in a footnote in section 3.3.

All the morphisms between the different curves are summarized in figure 1.2. All vertical maps and the two curved one are finite morphisms, while the straight diagonal maps are birational isomorphisms if the C_n are irreducible. Indeed, it is easily checked using the fiber product interpretation of $C_n^\#$ that the map $\tilde{C}_n \rightarrow C_n^\#$ is surjective, so that $C_n^\#$ is also irreducible. Since moreover the composite map $\tilde{C}_n \rightarrow C_n^\# \rightarrow C_n$ is a birational isomorphism and all curves are irreducible, both maps are birational isomorphisms.

1.3 Most important invariants of recursive towers

Our purpose is to study the following invariants of these towers.

At finite levels, for $n \geq 1$ and $r \geq 1$:

- the *arithmetic genus* γ_n of C_n and the *sharp-arithmetic genus* γ_n^\sharp of C_n^\sharp ;
- the common *geometric genus* g_n of C_n , C_n^\sharp and \widetilde{C}_n ;
- the number $N_r(\widetilde{C}_n) = \#\widetilde{C}_n(\mathbb{F}_{q^r})$ of \mathbb{F}_{q^r} -rational points of \widetilde{C}_n ;
- the number $B_r(\widetilde{C}_n)$ of points of \widetilde{C}_n of degree r .

Of course, for any $n \geq 1$, we have $g_n \leq \gamma_n^\sharp \leq \gamma_n$ and:

$$N_r(\widetilde{C}_n) = \sum_{d|r} r B_d(\widetilde{C}_n). \quad (2)$$

Ultimately, as usual, we introduce the two asymptotic invariants provided that the genus $\lim_{n \rightarrow \infty} g_n = \infty$:

$$\lambda_r(\mathcal{T}(X, \Gamma)) = \lim_{n \rightarrow +\infty} \frac{N_r(\widetilde{C}_n)}{g_n} \quad \text{and} \quad \beta_r(\mathcal{T}(X, \Gamma)) = \lim_{n \rightarrow +\infty} \frac{B_r(\widetilde{C}_n)}{g_n}.$$

Thanks to the well known lemma 1 below, these limit exist: we say, following [TV02], that a recursive tower is *asymptotically exact*. From equation (2), we have for any $r \geq 1$

$$\lambda_r(\mathcal{T}(X, \Gamma)) = \sum_{d|r} d \beta_d(\mathcal{T}(X, \Gamma)) \quad (3)$$

and the important inequality

$$A(q^r) \geq \lambda_r(\mathcal{T}(X, \Gamma)).$$

A recursive tower is interesting only if at least one λ_r exists and is non-zero, in which case the tower is said to be *good*. One can be more precise. It have been proved by Tsfasman that

$$\sum_{r=1}^{\infty} \frac{r \beta_r}{\sqrt{q^r} - 1} \leq 1,$$

generalizing the well known Drinfeld-Vladut bound. In [TV02], Tsfasman and Vladut defined the *deficiency* of an asymptotically exact tower by

$$\delta = 1 - \sum_{r=1}^{\infty} \frac{r \beta_r}{\sqrt{q^r} - 1} \in [0, 1]. \quad (4)$$

Of course, a tower is good if $\delta < 1$. It is said *optimal* if $\delta = 0$.

1.4 First easy necessary condition for a tower to be good

An irreducible *tower* is a sequence of absolutely irreducible curves $(X_n)_{n \geq 1}$ together with a family of finite dominant morphisms $X_{n+1} \rightarrow X_n$. It turns out that in the general context of smooth irreducible towers, Garcia and Stichtenoth have observed that the limits $\lambda_r(\mathcal{T}(X, \Gamma))$ and $\beta_d(\mathcal{T}(X, \Gamma))$ do exist for any $r \geq 1$, provided that the genus tends to infinity.

Lemma 1. (Garcia-Stichtenoth) *Let $(X_n)_{n \geq 1}$ be any irreducible tower of projective smooth absolutely irreducible curves, such that $\lim_{n \rightarrow \infty} g(X_n) = \infty$. Then the limits $\lambda_r((X_n)_{n \geq 1})$ and $\beta_r((X_n)_{n \geq 1})$ exist for any $r \geq 1$. This is the case for instance for smooth irreducible recursive towers $\tilde{\mathcal{T}}(X, \Gamma)$ such that $g(C_n) \rightarrow +\infty$.*

Proof — We begin by proving that the sequence $\frac{N_r(X_n)}{g(X_n)-1}$ decreases, hence converges. Indeed, we have, for any $n \in \mathbb{N}^*$, the inequalities $g(X_n) - 1 \geq [X_n, X_{n-1}](g(X_{n-1}) - 1)$ from Riemann-Hurwitz formula and $N_r(X_n) \leq [X_n, X_{n-1}]N_r(X_{n-1})$. Hence,

$$0 \leq \frac{N_r(X_n)}{g(X_n) - 1} \leq \frac{[X_n, X_{n-1}]N_r(X_{n-1})}{[X_n, X_{n-1}](g(X_{n-1}) - 1)} \leq \frac{N_r(X_{n-1})}{g(X_{n-1}) - 1}.$$

Since $(g(X_n))_{n \geq 1}$ goes to $+\infty$, it follows that $\lambda_r((X_n)_{n \geq 1})$ exists, hence also $\beta_r((X_n)_{n \geq 1})$ for any $r \geq 1$ by induction on r thanks to the relation (2). Now in case of irreducible recursive towers, since Γ contains no horizontal neither vertical components, π_1 and π_2 are both finite dominant morphisms hence by stability under base change each morphism $X_{n+1} \rightarrow X_n$ is finite dominant. \square

The reason why we assume in the setting section 1.1 that the correspondence is of type (d_1, d_2) with $d_1 = d_2$ is the following lemma.

Lemma 2. *Let (X, Γ) be a correspondence as in section 1.1, except that the type is assumed to be (d_1, d_2) . Let $\mathcal{T} = (C_n)_{n \geq 1}$ be the associated tower. Suppose that the curves C_n are irreducible for any $n \geq 1$, and that the geometric genus sequence $(g_n)_{n \geq 1}$ goes to infinity. If $d_1 \neq d_2$, then $\lambda_r(\mathcal{T}) = 0$ and $\beta_r(\mathcal{T}) = 0$ for any $r \geq 1$.*

Proof — Suppose for instance that $d_1 < d_2$, and let $r \geq 1$. Then one has $N_r(C_n) \leq N_r(C_1)d_1^{n-1}$. On the other hand, since the genus $g(C_n)$ goes to infinity, one can also suppose that $g(C_1) \geq 2$ and by Hurwitz genus formula, one has $g(C_n) - 1 \geq d_2^{n-1}(g(C_1) - 1)$ for any $n \geq 1$. Therefore $\lambda_r(\mathcal{T}(X, \Gamma)) = 0$ since $\left(\frac{d_1}{d_2}\right)^n \rightarrow 0$. The assertion for the β_r 's follows by induction from formula (3). \square

2 Genus sequences in a recursive tower

In order to compute the λ_r 's and β_r 's invariants of a recursive tower one needs to understand the behavior of the genus sequences. It turns out that g_n and $\gamma_n^\#$ are closely related thanks to adjunction formula (proposition 4 thanks to lemma 3), which leads us to theorem 5. The largest part of this section is devoted to the study of singular points of the singular tower $\mathcal{T}(X, \Gamma)$ (see proposition 6), and may be more importantly, to a minimal understanding of multiplicities in proposition 8 and its corollary which is useful as part of tools to prove theorem 24 at the end of the paper.

2.1 Arithmetic versus geometric genus in recursive towers

Let (X, Γ) be as in section 1.1 and consider $\mathcal{T}, \mathcal{T}^\sharp$ and $\tilde{\mathcal{T}}$ the associated towers of curves. The sharp model turns here to be a useful tool to understand the geometric genus sequence $(g_n)_{n \geq 1}$. We proceed in two steps: first we compare the geometric genus g_n with the arithmetic genus γ_n^\sharp using the adjunction formula on the smooth surface $\tilde{C}_{n-1} \times X$, then we prove an induction relation between g_n and g_{n-1} involving terms coming from desingularization of C_n^\sharp . This relation permits us to give a closed formula for the genus sequence for the tower in [BS07] in section 3.3.

- The first step is classical. For any $n \geq 2$, and any $P \in C_n^\sharp(\overline{\mathbb{F}_q})$ be a geometric point, let δ_P denote the *measure of the singularity* at P (see Hartshorne [Har77], Chap IV, Ex 1.8 or Liu [Liu02], §7.5), that is¹

$$\delta_P = \dim_{\overline{\mathbb{F}_q}} \tilde{\mathcal{O}}_P / \mathcal{O}_P,$$

where \mathcal{O}_P and $\tilde{\mathcal{O}}_P$ denote the local ring of C_n^\sharp at P and its integral closure. This measure is non-zero if and only if the point P is singular so it makes sense to define

$$\Delta_n = \sum_{P \in C_n^\sharp(\overline{\mathbb{F}_p})} \delta_P \quad (5)$$

as a measure of the whole singularities of C_n^\sharp . Then the geometric and arithmetic genus of C_n^\sharp are related by

$$\gamma_n^\sharp = g_n + \Delta_n \quad (6)$$

(see loc. cit.). In section 3.3, we illustrate, on a non trivial example, the fact that the Δ_n 's can be computed.

- Second, to prove the induction relation, we need the following lemma.

Lemma 3. *Let $f_i : Y_i \rightarrow X_i$ be finite morphisms of smooth absolutely irreducible projective curves of degree n_i for $i = 1, 2$, and let $F : Y_1 \times Y_2 \rightarrow X_1 \times X_2$ be the product morphism $F = f_1 \times f_2$. If Γ is a correspondence of type (d_1, d_2) on $X_1 \times X_2$, then the arithmetic genus $\gamma(F^*(\Gamma))$ of the pull-back $F^*(\Gamma)$ of Γ by F is given by*

$$2\gamma(F^*(\Gamma)) - 2 = n_1 n_2 \Gamma^2 + n_2 d_2 (2g(Y_1) - 2) + n_1 d_1 (2g(Y_2) - 2)$$

where $g(Y_i)$ denotes the genus of Y_i ($i = 1, 2$) and where Γ^2 is the self-intersection of Γ computed in the group $\text{NS}(X_1 \times X_2)$.

Proof — By adjunction formula (see Hartshorne [Har77], Chap V, Prop 1.5), the arithmetic genus is given by

$$2\gamma(F^*(\Gamma)) - 2 = F^*(\Gamma) \cdot (F^*(\Gamma) + K_{Y_1 \times Y_2}),$$

where $K_{Y_1 \times Y_2}$ is the canonical class in the Neron-Severi group $\text{NS}(Y_1 \times Y_2)$ of the smooth surface $Y_1 \times Y_2$. This class $K_{Y_1 \times Y_2}$ is known to be $(2g(Y_2) - 2)H + (2g(Y_1) - 2)V$ where H and V denote the horizontal and vertical classes in $\text{NS}(Y_1 \times Y_2)$. Then

$$2\gamma(F^*(\Gamma)) - 2 = F^*(\Gamma) \cdot F^*(\Gamma) + (2g(Y_2) - 2)F^*(\Gamma) \cdot H + (2g(Y_1) - 2)F^*(\Gamma) \cdot V.$$

¹Consistency would require sharp exponents for the following δ_P , \mathcal{O}_P and Δ_n . For simplicity, we choose to drop them.

Denote by h and v the horizontal and vertical classes in $\text{NS}(X_1 \times X_2)$. By the projection formula, we have

$$\begin{aligned} F^*(\Gamma) \cdot F^*(\Gamma) &= \Gamma \cdot F_* F^*(\Gamma) = \Gamma \cdot \deg(F) \Gamma = n_1 n_2 \Gamma^2, \\ F^*(\Gamma) \cdot H &= \Gamma \cdot F_*(H) = \Gamma \cdot n_1 h = n_1 d_1, \\ F^*(\Gamma) \cdot V &= \Gamma \cdot F_*(V) = \Gamma \cdot n_2 v = n_2 d_2 \end{aligned}$$

since we have $d_1 = \Gamma \cdot h$ and $d_2 = \Gamma \cdot v$ by the very definition of the type (d_1, d_2) . \square

Proposition 4. *Let (X, Γ) be a correspondence as in section 1.1. Let $(g_n)_{n \geq 1}$ and $(\gamma_n^\sharp)_{n \geq 1}$ be the geometric and sharp-arithmetic genus sequence of the associated tower.*

(i) *The sharp-arithmetic genus γ_n^\sharp and the geometric genus g_{n-1} are related, for $n \geq 2$, by*

$$\gamma_n^\sharp - 1 = d(g_{n-1} - 1) + d^{n-2} \left[(\gamma_2^\sharp - 1) - d(g_1 - 1) \right].$$

(ii) *For any $n \geq 1$, the geometric genus g_n is given by*

$$g_n - 1 = \begin{cases} (n-1)d^{n-2} [(\gamma_2 - 1) - d(g_1 - 1)] + d^{n-1}(g_1 - 1) - \sum_{i=2}^n d^{n-i} \Delta_i & (\text{general case}) \\ (n-1)d^{n-2} [(g_2 - 1) - d(g_1 - 1)] + d^{n-1}(g_1 - 1) & (\text{smooth case}) \end{cases}$$

where the Δ_i 's are defined in formula (5) and where γ_2 denote the arithmetic genus of Γ .

Proof — To prove (i), we first apply lemma 3 with $Y_1 = \tilde{C}_{n-1}$, $Y_2 = X$, $X_1 = X_2 = X$, $f_1 = \pi_{n-1}^{n-1} \circ \nu_{n-1}$ (see §1.2 for definitions) and $f_2 = \text{Id}$. We get $n_1 = d^{n-2}$, $n_2 = 1$ and

$$2\gamma_n^\sharp - 2 = d^{n-2} \Gamma^2 + d(2g_{n-1} - 2) + d^{n-1}(2g_1 - 2).$$

In particular, for $n = 2$, this leads to $\Gamma^2 = (2\gamma_2 - 2) - 2d(2g_1 - 2)$. Substituting this expression of Γ^2 in the preceding equation permits to conclude.

To prove (ii), let $u_n = \frac{g_n - 1}{d^n}$. From (i) together with (6), we deduce the induction relation

$$u_n = u_{n-1} + \frac{(g_2 - 1) - d(g_1 - 1) + \Delta_2}{d^2} - \frac{\Delta_n}{d^n}.$$

An easy calculation gives the general formula. If all the C_n are smooth, then all Δ_i vanish and $\gamma_2 = g_2$. \square

2.2 Another necessary condition for a tower to be good

We prove that under the irreducibility assumption, the tower need either to be singular, either to be constructed from an étale correspondence over a general curve X in order to be interesting.

Proposition 5. *Let (X, Γ) be a correspondence as in section 1.1 and let $\mathcal{T} = (C_n)_{n \geq 1}$ be the associated recursive tower. Suppose that C_n is irreducible for any $n \geq 1$ and that the sequence genus $(g_n)_{n \geq 1}$ tends to infinity. If there is at least one $r \geq 1$ such that $\lambda_r(\mathcal{T}) > 0$, then*

(i) *either C_n is singular for any n greater than some n_0 ;*

(ii) *either $g_1 = g(X) \geq 2$ and both covers $\pi_i : \Gamma \rightarrow X$ for $i = 1, 2$ are étale over X .*

Proof — Suppose that C_n is smooth for any $n \geq 1$. Then by the last item of proposition 4, one obtain for any $n \geq 1$

$$g_n = (n-1)d^{n-2}[(g_2-1) - d(g_1-1)] + d^{n-1}(g_1-1) + 1.$$

If $(g_2-1) - d(g_1-1) \neq 0$, then

$$g_n \sim (n-1)((g_2-1) - d(g_1-1))d^{n-2}.$$

On the other hand, using the projection morphism from C_n to C_1 given by $(P_1, \dots, P_n) \mapsto P_1$ of degree d^{n-1} , one deduce that

$$N_r(C_n) \leq N_r(C_1) \times d^{n-1}$$

for any $r \geq 1$. Therefore,

$$\frac{N_r(C_n)}{g_n} \leq \frac{N_r(C_1)d^{n-1}}{g_n} \sim \frac{N_r(C_1)}{(g_2-1) - d(g_1-1)} \times \frac{d^{n-1}}{(n-1)d^{n-2}} \xrightarrow{n \rightarrow +\infty} 0$$

and $\lambda_r(\mathcal{T}(X, \Gamma)) = 0$.

If $(g_2-1) - d(g_1-1) = 0$, then both projections must be étale and one must have $g_1 = g(X) \geq 1$. Finally if X is an elliptic curve, then Riemann-Hurwitz yields to $g_n = 1$ for any n , which doesn't grows to infinity. \square

It worth to notice that if both morphisms are non-étale, then not only the tower $(C_n)_n$ need to be singular, but it needs to be sufficiently singular in the sense that that $g_n = o(\gamma_n)$. More precisely, the number of rational points of \tilde{C}_n is at most $\#X(\mathbb{F}_q) \times d^n$, while in the smooth case we have seen that its genus is about $\gamma_n = g_n \sim cst.nd^n$. Hence, in the singular case, the geometric genus

$$g_n = \gamma_n^\sharp - \Delta_n$$

have to drop from an arithmetic genus γ_n to sharp arithmetic genus $\gamma_n^\sharp \sim cst.nd^n$ to at most $cst \times d^n$ in order to get at least one non-zero λ_r . This means that the contribution of singularities Δ_n need to be equivalent to $\gamma_n^\sharp \sim cst \times nd^n$.

This proposition 5 motivates a more accurate study of singular points of C_n . This is the aim of the next section.

2.3 Singular points of C_n

The goal of this section is twofold. First, we characterize the singular points of the curves C_n in prop 6. Then we prove, for later use, proposition 8 about the singularity of cycles.

To begin with, let X and Y be two smooth projective absolutely irreducible curves over² \mathbb{F}_q and let Γ be a correspondence on $X \times Y$, without any vertical, nor horizontal, components. We still denote by π_1 and π_2 the projections onto the first and second factors. Let $(P, Q) \in \Gamma$ be a geometric point and consider affine neighborhoods of $P \in U \subset \mathbb{A}^r$, $Q \in V \subset \mathbb{A}^s$ and $(P, Q) \in W \subset \mathbb{A}^{r+s}$. Suppose that the three affine curves U , V and W are respectively defined by $(r-1) + \rho$, $(s-1) + \sigma$ and $1 + \tau$ equations in addition of the equations coming from those of U and V (where $r, s \geq 1$ and $\rho, \sigma, \tau \geq 0$). Taking into account that the equations

²In fact, a large part of this section works over an arbitrary field k .

defining U (resp. V) only depend on the r first (resp. s last) indeterminates, we deduce that the jacobian matrix of the point $(P, Q) \in W$ has the following shape:

$$J_{\Gamma}(P, Q) = \begin{pmatrix} \boxed{J_X(P)} & & \\ \boxed{A} & \boxed{B} & \\ & & \boxed{J_Y(Q)} \end{pmatrix} \quad (7)$$

where $J_X(P)$ and $J_Y(Q)$ denote the jacobian matrices of X at P and Y at Q . Since the curves X and Y are supposed to be smooth at P and Q , the jacobian submatrices $J_X(P)$ and $J_Y(Q)$ have rank equal to $(r - 1)$ and $(s - 1)$ respectively. Therefore, due to its shape, $J_{\Gamma}(P, Q)$ has rank greater than $r + s - 2$. On the other hand, $J_{\Gamma}(P, Q)$ has rank less than $r + s - 1$ since Γ is a curve locally embedded in \mathbb{A}^{r+s} . We easily deduce that

$$\text{rk}(J_{\Gamma}(P, Q)) = r + s - 1 \iff \text{rk} \begin{pmatrix} J_X(P) \\ A \end{pmatrix} = r \text{ or } \text{rk} \begin{pmatrix} B \\ J_X(P) \end{pmatrix} = s \quad (8)$$

$$\text{rk}(J_{\Gamma}(P, Q)) = r + s - 2 \iff \text{rk} \begin{pmatrix} J_X(P) \\ A \end{pmatrix} = r - 1 \text{ and } \text{rk} \begin{pmatrix} B \\ J_X(P) \end{pmatrix} = s - 1 \quad (9)$$

Study of the smoothness of Γ at (P, Q) . The point (P, Q) is smooth if and only if the jacobian matrix has maximal rank, that is by (8) and (9):

$$\begin{array}{l} \text{the point } (P, Q) \in \Gamma \\ \text{is } \mathbf{singular} \end{array} \iff \text{rk} \begin{pmatrix} J_X(P) \\ A \end{pmatrix} = r - 1 \text{ and } \text{rk} \begin{pmatrix} B \\ J_X(P) \end{pmatrix} = s - 1 \quad (10)$$

$$\begin{array}{l} \text{the point } (P, Q) \in \Gamma \\ \text{is } \mathbf{smooth} \end{array} \iff \text{rk} \begin{pmatrix} J_X(P) \\ A \end{pmatrix} = r \text{ or } \text{rk} \begin{pmatrix} B \\ J_X(P) \end{pmatrix} = s \quad (11)$$

Suppose the former, that is $(P, Q) \in \Gamma$ smooth. Then one can extract from $J_X(P)$ (resp. from $J_Y(Q)$) an $(r - 1) \times r$ (resp. an $(s - 1) \times s$) block $J'_X(P)$ (resp. $J'_Y(Q)$) of maximal rank and from the "correspondence" block-line AB exactly one line such that the matrix

$$J'_{\Gamma}(P, Q) = \begin{pmatrix} \boxed{J'_X(P)} & & \\ a_1 \dots \dots \dots a_r & b_1 \dots \dots \dots b_s & \\ & & \boxed{J'_Y(Q)} \end{pmatrix}$$

has maximal rank equal to $(r+s-1)$. Let $\delta_1(J'_X(P)), \dots, \delta_r(J'_X(P))$ and $\delta_1(J'_Y(Q)), \dots, \delta_s(J'_Y(Q))$ be the minors of $J'_X(P)$ and $J'_Y(Q)$ (listed with alternate signs), and define

$$\pi'_2(P, Q) = \begin{vmatrix} & J'_X(P) & \\ a_1 & \dots & a_r \end{vmatrix} \quad \text{and} \quad \pi'_1(P, Q) = \begin{vmatrix} b_1 & \dots & b_s \\ & J'_Y(Q) & \end{vmatrix}. \quad (12)$$

Then the maximal minors of $J'_\Gamma(P, Q)$ are the $\delta_i(J'_X(P))\pi'_1(P, Q)$ for $1 \leq i \leq r$ and the $\delta_j(J'_Y(Q))\pi'_2(P, Q)$ for $1 \leq j \leq s$. Moreover its kernel, which is nothing else than the tangent line of Γ at (P, Q) , is thus generated by the vector³

$$\begin{pmatrix} \delta_1(J'_X(P))\pi'_1(P, Q) \\ \vdots \\ \delta_r(J'_X(P))\pi'_1(P, Q) \\ \delta_1(J'_Y(Q))\pi'_2(P, Q) \\ \vdots \\ \delta_s(J'_Y(Q))\pi'_2(P, Q) \end{pmatrix}.$$

Because at least one of the minors of $J'_X(P)$ and of $J'_Y(Q)$ are non zero by smoothness of X and Y at P and Q , this vector is non zero if and only if $\pi'_2(P, Q)$ or $\pi'_1(P, Q)$ are non zero, that is if and only if Γ is smooth at (P, Q) by (11).

Note for later use that $\pi'_2(P, Q) = 0$ (resp. $\neq 0$) if and only if $\text{rk} \left(\begin{smallmatrix} J_X(P) \\ A \end{smallmatrix} \right) = r - 1$ (resp. $= r$). Hence this vanishing doesn't depends on the choice of the line $a_1, \dots, a_r, b_1, \dots, b_s$ in the line-block AB . Of course this holds also for the vanishing of $\pi'_1(P, Q)$.

Study of etaleness of π_i at $(P, Q) \in \Gamma$. Since Q is a smooth point of Y , the projection $\pi_2 : \Gamma \rightarrow Y$ is étale at (P, Q) if and only if the point $(P, Q) \in \Gamma$ is smooth and the induced map $\text{Ker}(J_\Gamma(P, Q)) \rightarrow \text{Ker}(J_Y(Q))$ is an isomorphism, that is non zero since kernels are lines here. In view of the generator of the tangent line at (P, Q) , this application is non zero if and only if $\pi'_2(P, Q)$ is non zero because at least one of the minors of $J'_Y(Q)$ is non zero. Thanks to the remark at the end of the study of smoothness, we deduce that

$$\begin{array}{l} \pi_2 \text{ is étale} \\ \text{at } (P, Q) \end{array} \Leftrightarrow \text{rk}(J_\Gamma(P, Q)) = r + s - 1 \text{ and } \text{rk} \left(\begin{smallmatrix} J_X(P) \\ A \end{smallmatrix} \right) = r \quad \stackrel{\text{by (8)}}{\Leftrightarrow} \quad \text{rk} \left(\begin{smallmatrix} J_X(P) \\ A \end{smallmatrix} \right) = r. \quad (13)$$

Of course, we also prove the same way that

$$\begin{array}{l} \pi_1 \text{ étale} \\ \text{at } (P, Q) \end{array} \Leftrightarrow \text{rk}(J_\Gamma(P, Q)) = r + s - 1 \text{ and } \text{rk} \left(\begin{smallmatrix} B \\ J_Y(Q) \end{smallmatrix} \right) = s \quad \stackrel{\text{by (8)}}{\Leftrightarrow} \quad \text{rk} \left(\begin{smallmatrix} B \\ J_Y(Q) \end{smallmatrix} \right) = s. \quad (14)$$

From (10), (13) and (14), we deduce that the singularity of (P, Q) in Γ can be characterized only using étaleness by

$$\begin{array}{l} \text{the point } (P, Q) \in \Gamma \\ \text{is **singular**} \end{array} \iff \begin{array}{l} \text{both projections } \pi_1 \text{ and } \pi_2 \\ \text{are **not** étale at } (P, Q) \end{array}. \quad (15)$$

In the following proposition, we prove that such a characterization still occurs for the curves C_n of a recursive tower.

Proposition 6. *Let (X, Γ) be a correspondence as in section 1.1 and let $(C_n)_{n \geq 1}$ be the associated recursive tower. A point $(P_1, \dots, P_n) \in C_n$ is singular if and only if there exist $1 \leq i \leq j < n$ such that π_2 is not étale at (P_i, P_{i+1}) and π_1 is not étale at (P_j, P_{j+1}) .*

Proof — One can suppose that the points $P_1, \dots, P_n \in X$ are contained in the same affine open subspace $U \subset \mathbb{A}^r$, and that the affine curve U is defined by $(r - 1) + \rho$ equations. Suppose also that, locally in $U \times U \subset \mathbb{A}^{2r}$, Γ is defined, in addition of the equations coming from U , by $1 + \tau$

³ Recall that if $M \in M_{r-1, r}(k)$ is a matrix of rank $r - 1$, then its kernel is generated by the vector whose coordinates are its minors with alternate signs

equations in $\mathbb{A}^r \times \mathbb{A}^r = \mathbb{A}^{2r}$. Thus C_n is locally embedded in \mathbb{A}^{nr} and the jacobian matrix of C_n at (P_1, \dots, P_n) is an $((nr - 1) + n\rho + (n - 1)\tau) \times nr$ matrix looking like

$$J_{C_n}(P_1, \dots, P_n) = \begin{pmatrix} \boxed{J_X(P_1)} & & & & \\ \boxed{A_1} & \boxed{B_1} & & & \\ & \boxed{J_X(P_2)} & & & \\ & & \ddots & & \\ & & & \boxed{A_{n-1}} & \boxed{B_{n-1}} \\ & & & & \boxed{J_X(P_n)} \end{pmatrix}$$

Since every $P_i \in X$ is smooth, every "jacobian" block has rank equal to $(r - 1)$. Since Γ is locally a curve in \mathbb{A}^{2r} , every block looking like the jacobian matrix of equation (7) has rank less than $(2r - 1)$ (and grater than $(2r - 2)$). Hence the rank of $J_{C_n}(P_1, \dots, P_n)$ is grater than $n(r - 1)$ (contributions of the "jacobian" blocks) and every "correspondence"-blocks have a contribution to the whole rank at most 1.

A point $(P_1, \dots, P_n) \in C_n$ is smooth if and only if the rank of $J_{C_n}(P_1, \dots, P_n)$ is equal to $nr - 1 = n(r - 1) + (n - 1)$. This is possible only if each of the $(n - 1)$ "correspondence"-blocks has a contribution to the whole rank exactly equal to 1. Conversely, a point $(P_1, \dots, P_n) \in C_n$ is singular if and only if there exists at least one "correspondence"-block whose lines are all in the vector space generated by the remaining lines. In particular⁴, if $(P_1, \dots, P_m) \in C_m$ is singular then so is $(P_1, \dots, P_m, P_{m+1}, \dots, P_n) \in C_n$ for every $n \geq m$. Then $(P_1, \dots, P_n) \in C_n$ is singular if and only if there exists $1 \leq j \leq n - 1$ such that $(P_1, \dots, P_j) \in C_j$ is smooth while $(P_1, \dots, P_{j+1}) \in C_{j+1} \subset C_j \times X$ is singular. Applying the beginning of this section to the correspondence C_{j+1} on the product $C_j \times X$ of smooth curves, we deduce that this is equivalent to

$$\text{rk} \left(\begin{pmatrix} \boxed{J_{C_j}(P_1, \dots, P_j)} \\ \boxed{A_j} \end{pmatrix} \right) = rj - 1 \quad \text{and} \quad \text{rk} \left(\begin{pmatrix} B_j \\ J_X(P_{j+1}) \end{pmatrix} \right) = r - 1,$$

where A_j and B_j denote the two blocks coming from the condition $(P_j, P_{j+1}) \in \Gamma$. By the negation of (14), the second equality occurs if and only if the projection π_1 is not étale at (P_j, P_{j+1}) . As to the first equality, it occurs if and only if there exists $1 \leq i \leq j$ such that π_2 is not étale at (P_i, P_{i+1}) , that is

$$\exists 1 \leq i \leq j, \quad \text{rk} \left(\begin{pmatrix} J_X(P_i) \\ A_i \end{pmatrix} \right) = r - 1.$$

Indeed, since $(P_1, \dots, P_j) \in C_j$ is smooth, the jacobian matrix $J_{C_j}(P_1, \dots, P_j)$ has a rank equal to $rj - 1$. As in the case of a correspondence on a product of only two curves, one can extract $(r - 1)$ -rank blocks $J'_X(P_1), \dots, J'_X(P_j)$ from the "jacobian" blocks $J_X(P_1), \dots, J_X(P_j)$, and lines $(a_{k,1}, \dots, a_{k,r}, b_{k,1}, \dots, b_{k,r})$ from the "correspondence" block (A_k, B_k) for $1 \leq k \leq j - 1$, to obtain a $(rj - 1) \times rj$ matrix $J'_{C_j}(P_1, \dots, P_j)$ of maximal rank. The first rank equality is thus equivalent to the fact that for every choice of line $(a_{j,1}, \dots, a_{j,r})$ in the block A_j , this line must be a linear combination of the lines of $J'_{C_j}(P_1, \dots, P_j)$. So one must have

$$\begin{vmatrix} J'_X(P_1) & & \\ a_{1,1} & \cdots & a_{1,r} \end{vmatrix} \times \cdots \times \begin{vmatrix} J'_X(P_{j-1}) & & \\ a_{j-1,1} & \cdots & a_{j-1,r} \end{vmatrix} \times \begin{vmatrix} J'_X(P_j) & & \\ a_{j,1} & \cdots & a_{j,r} \end{vmatrix} = 0$$

⁴This is a particular feature of recursive towers since it doesn't hold true in general that if Γ is a correspondence in $X \times X$, if $\pi : Y \rightarrow X$ is a morphism from a singular curve Y (here $Y = C_m$) and if P is a singular point in Y such that $(\pi(P), Q) \in \Gamma$, then (P, Q) is singular in the pullback curve $(\pi \times Id)^*(\Gamma)$.

Hence,

$$\text{either } \text{rk} \begin{pmatrix} J_X(P_j) \\ A_j \end{pmatrix} = r - 1 \quad \text{either} \quad \exists 1 \leq i \leq j - 1, \quad \begin{vmatrix} J'_X(P_i) & & \\ & \ddots & \\ a_{i,1} & & a_{i,r} \end{vmatrix} = 0$$

But, as already noted at the end of the paragraph dealing with the smoothness of point in $\Gamma \subset X \times Y$, the last case is equivalent to

$$\text{rk} \begin{pmatrix} J_X(P_i) \\ A_i \end{pmatrix} = r - 1.$$

In both cases, by the negation of (13), we conclude that there exists $1 \leq i \leq j$ such that the projection π_2 is not étale at (P_i, P_{i+1}) . \square

In the case of correspondences of type $\Gamma_{f,g}$ on $X = \mathbb{P}^1$ so widely used in the literature, one can take $r = 1$ and $\rho = \tau = 0$ in the above proof. The characterization of the singular points becomes:

Corollary 7. *Let $\Gamma_{f,g}$ be a correspondence on $\mathbb{P}^1 \times \mathbb{P}^1$ where f and g are two non constant functions on \mathbb{P}^1 and let $(C_n)_{n \geq 1}$ be the corresponding singular recursive tower. A point $(P_1, \dots, P_n) \in C_n$ is singular if and only if there exist $1 \leq i < j \leq n$ such that P_i is a ramified point of f and P_j is a ramified point of g .*

Proof — In this context, all the jacobian blocks are empty, and the correspondence blocks can always be reduced to a single line. The étaleness of the projection π_2 (resp. π_1) at a point $(P, Q) \in \Gamma_{f,g}$ becomes $f'(P) \neq 0$ (resp. $g'(Q) \neq 0$). The corollary follows. \square

We will need at the end of this paper a characterization of the singular points contained in the intersection of the curve C_n and the hypersurface $P_1 = P_n$ in X^n . In the following proposition, we continue to make use of a local embedding of X in \mathbb{A}^r and of the associated determinants $\pi'_1(P, Q)$ and $\pi'_2(P, Q)$ defined in (12).

Proposition 8. *Let (X, Γ) be a correspondence as in section 1.1, let $(C_n)_{n \geq 1}$ be the associated recursive tower, and let H_n denote the hypersurface of X^n defined by $P_1 = P_n$.*

Consider $(P_1, \dots, P_n) \in C_n$ a smooth point. Then it is singular in the intersection $C_n \cap H_n$ if and only if

$$(-1)^{r(n-1)} \prod_{i=1}^{n-1} \pi'_2(P_i, P_{i+1}) = \prod_{i=1}^{n-1} \pi'_1(P_i, P_{i+1}), \quad \text{in } \mathbb{F}_q(P_1, \dots, P_n),$$

where, π'_1 and π'_2 are the determinants defined by (12).

Proof — We work in the affine neighborhood \mathbb{A}^{rn} of (P_1, \dots, P_n) . If $(P_1, \dots, P_n) \in C_n \cap H_n$, that is if $P_1 = P_n$, then

$$J_{C_n \cap H_n}(P_1, \dots, P_n) = \left(\begin{array}{c} \boxed{J_{C_n}(P_1, \dots, P_n)} \\ \boxed{I_r} \cdots \cdots \cdots \boxed{-I_r} \end{array} \right)$$

where I_r is the identity matrix of size r . Since $(P_1, \dots, P_n) \in C_n$ is assumed to be smooth, the jacobian matrix $J_{C_n}(P_1, \dots, P_n)$ has rank equal to $nr - 1$. As to the point $(P_1, \dots, P_n) \in C_n \cap H_n$,

it is singular if and only if the matrix $J_{C_n \cap H_n}(P_1, \dots, P_n)$ is still of rank $rn - 1$, if and only if the r -th last lines of $J_{C_n \cap H_n}(P_1, \dots, P_n)$ lie in the vector space generated by the lines of $J_{C_n}(P_1, \dots, P_n)$.

As in the proof of proposition 6, after cancellation of redundant lines of the jacobian matrix $J_{C_n}(P_1, \dots, P_n)$, we obtain a $(rn - 1) \times rn$ matrix $J'_{C_n}(P_1, \dots, P_n)$ of maximal rank. The singularity conditions then reduce to the vanishing of the r determinants

$$\left| \begin{array}{c} \boxed{J'_{C_n}(P_1, \dots, P_n)} \\ 1, 0, \dots, 0 \dots\dots -1, 0, \dots, 0 \end{array} \right| = \dots = \left| \begin{array}{c} \boxed{J'_{C_n}(P_1, \dots, P_n)} \\ 0, \dots, 0, 1 \dots\dots 0, \dots, 0, -1 \end{array} \right| = 0.$$

For each i , $1 \leq i \leq r$, expanding the determinant along the last line leads, since $P_1 = P_n$, to

$$(-1)^{rn+i} \delta_i(J_X(P_1)) \prod_{i=1}^{n-1} \pi'_1(P_i, P_{i+1}) + (-1)^{rn+r(n-1)+i} \times (-1) \delta_i(J_X(P_1)) \prod_{i=1}^{n-1} \pi'_2(P_i, P_{i+1}) = 0.$$

Since $P_1 \in X$ is smooth, at least one of the $\delta_i(J_X(P_1))$'s, $1 \leq i \leq r$, is non-zero and the result follows. \square

Corollary 9. *With the hypothesis of the preceding proposition, let (P_1, \dots, P_l, P_1) be a smooth cycle of length $l \geq 1$ in C_{l+1} . Then there exists an iteration ρ of this cycle that becomes singular in $C_{\rho l+1} \cap H_{\rho l+1}$.*

Proof. Applying the preceding proposition, we know that the ρ -th iterate cycle is singular if and only if

$$(-1)^{r\rho l} \prod_{i=1}^l \pi'_2(P_i, P_{i+1})^\rho = \prod_{i=1}^l \pi'_1(P_i, P_{i+1})^\rho.$$

Hence ρ equal to $\#\mathbb{F}_q(P_1, \dots, P_l) - 1$ works. \square

3 Graphs and recursive towers

3.1 The geometric graph and its arithmetic and singular subgraphs

To a recursive tower $\mathcal{T}(X, \Gamma)$ given by an irreducible correspondence Γ on $X \times X$, we associate in a very natural way an incidence “geometric” graph whose vertices are the geometric points of X and whose edges depend on Γ . Some of its “arithmetic” subgraphs will play a crucial role till the end of this paper and in the proof of theorem 24.

Definition 10. *Let (X, Γ) be a correspondence as in section 1.1.*

- (i) The **geometric graph** $\mathcal{G}_\infty(X, \Gamma) = \mathcal{G}_\infty$ is the graph whose vertices are the geometric points of X , and for which there is an oriented edge from $P \in X(\overline{\mathbb{F}}_q)$ to $Q \in X(\overline{\mathbb{F}}_q)$ if $(P, Q) \in \Gamma(\overline{\mathbb{F}}_q)$.
- (ii) An oriented edge $P \rightarrow Q$ of the graph is said to be *étale* by π_1 if the morphism π_1 is étale at (P, Q) . In the same way, the edge $P \rightarrow Q$ is said to be *étale* by π_2 if the morphism π_2 is étale at (P, Q) . The **singular part** of the graph \mathcal{G}_∞ , denoted by $\mathcal{G}_{\text{sing}}$, is the union of all weakly connected components⁵ containing at least one edge which is not étale by π_1 or by π_2 .

⁵ A directed graph is called weakly connected if replacing all of its directed edges with undirected edges produces a connected (undirected) graph. A weakly connected component is a maximal weakly connected subgraph.

- (iii) For any subset $S \subset X(\overline{\mathbb{F}}_q)$, the graph \mathcal{G}_S is the subgraph of \mathcal{G}_∞ , whose vertices are the points of S and where there is an oriented edge from $P \in S$ to $Q \in S$ if there is one in \mathcal{G}_∞ , that is if $(P, Q) \in \Gamma(\overline{\mathbb{F}}_q)$.
- (iv) In particular for $S = X(\mathbb{F}_{q^r})$, $1 \leq r < +\infty$, we denote by \mathcal{G}_r the subgraph $\mathcal{G}_{X(\mathbb{F}_{q^r})}$ and we call it the ***r-th arithmetic graph***.

This graph is a convenient way to “see” some of the most important features of a recursive tower:

- The geometric points of C_n are in bijection with the paths of length $n - 1$ of \mathcal{G}_∞ (that is n vertices and $n - 1$ edges) while the arithmetic points defined over \mathbb{F}_{q^r} are in bijection with the paths of length $n - 1$ of \mathcal{G}_r .
- The non étale points $(P, Q) \in \Gamma$ can be read off the in and out degrees of the graph \mathcal{G}_∞ . Indeed, for every vertex $P \in X(\overline{\mathbb{F}}_q)$, the out-degree $d^+(P)$ (resp. in-degree $d^-(P)$) at P of the graph \mathcal{G}_∞ is equal to d *except* if there exists at least one point $(P, Q) \in \Gamma$ (resp. $(Q, P) \in \Gamma$) above P which is not étale by π_1 (resp. π_2), in which case this out (resp. in) degree is $< d$.
- The complementary part of the singular part of \mathcal{G}_∞ is a d -regular graph in the graph theoretic sense, which means that at every vertex the out and the in degrees are equal to d .

One can even be more precise.

Proposition 11. *Let (X, Γ) be a correspondence as in section 1.1.*

- (i) *A path of length $(n - 1)$ in \mathcal{G}_∞ corresponds to a singular point of C_n if and only if there exist $1 \leq i \leq j < n$ such that the edge $P_i \rightarrow P_{i+1}$ is not étale by π_2 and the edge $P_j \rightarrow P_{j+1}$ is not étale by π_1 . In particular, this path is contained in the singular part $\mathcal{G}_{\text{sing}}$ of \mathcal{G}_∞ .*
- (ii) *Every path of length $(n - 1)$ outside the singular part corresponds to a smooth point of C_n .*

Proof — The first item is only a translation of proposition 6 whereas the second one follows trivially. \square

For instance, we represent in figure 3.1 the second arithmetic graph \mathcal{G}_2 for the very nice tame tower of [GSR03] recursively defined by $y^2 = \frac{x^2+1}{2x}$ over \mathbb{F}_5 . The singular part $\mathcal{G}_{\text{sing}}$ is the subgraph whose vertices are the points of $\mathbb{P}^1(\mathbb{F}_5) = \{0, \pm 1, \pm 2, \infty\}$.

3.2 Finite complete sets and rational points

Definition 12. *A subset S of $X(\overline{\mathbb{F}}_q)$ is said to be:*

- (i) ***forward complete*** if every point of S has all its outgoing neighbors in \mathcal{G}_∞ inside S , that is if $\pi_2(\pi_1^{-1}(S)) \subset S$;
- (ii) ***backward complete*** if every point of S has all its incoming neighbors in \mathcal{G}_∞ inside S , that is if $\pi_1(\pi_2^{-1}(S)) \subset S$;
- (iii) ***complete*** if it is both backward and forward complete.

Remark – Being complete for a subset $S \subset X(\overline{\mathbb{F}}_q)$ does **NOT** mean that the graph \mathcal{G}_S is complete in the usual sense of graph theory.

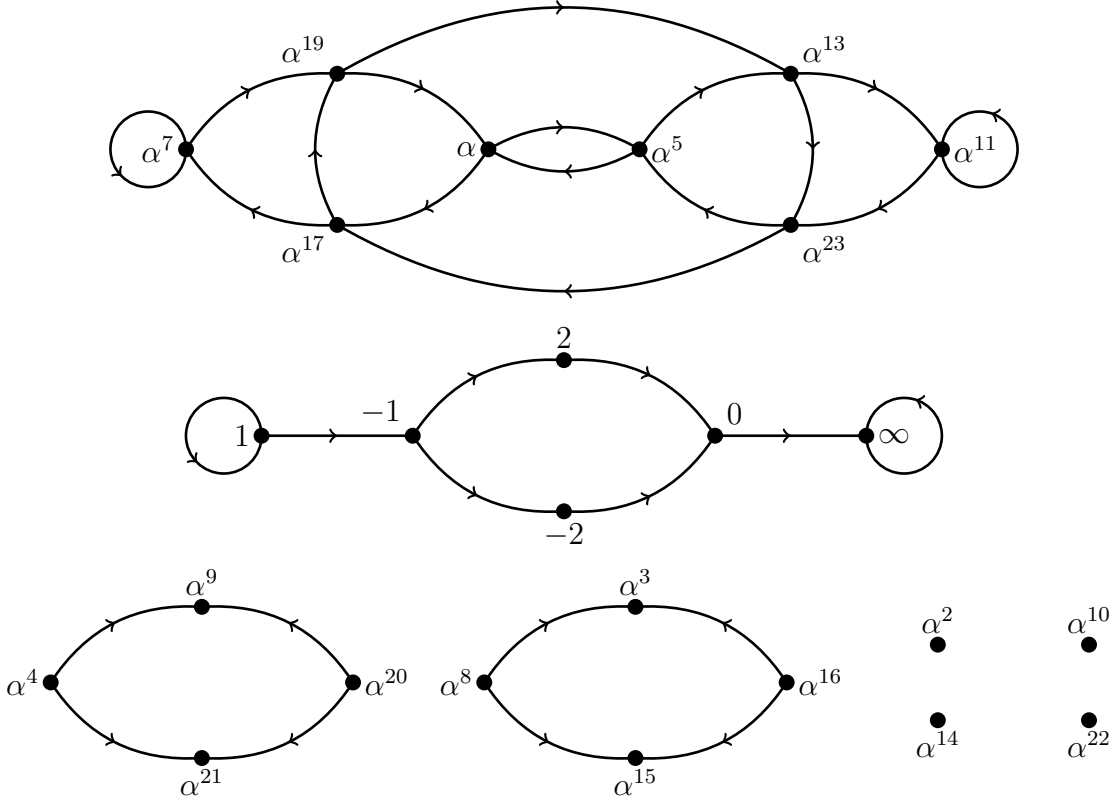


Figure 2: The second arithmetic graph $\mathcal{G}_2 \left(\mathbb{P}^1, \frac{x^2+1}{2x} = y \right)$ over \mathbb{F}_5

A subgraph \mathcal{G}_S outside \mathcal{G}_{sing} is complete in our sense if and only if it is d -regular in standard graph theory. The following examples will illustrate this.

In the example of figure 3.1, the sets $\{\alpha, \alpha^5, \alpha^7, \alpha^{11}, \alpha^{13}, \alpha^{17}, \alpha^{19}, \alpha^{23}\}$ and $\{0, \pm 1, \pm 2, \infty\}$ are complete while the set $\{\alpha^4, \alpha^9, \alpha^{20}, \alpha^{21}\}$ is neither forward neither backward complete. The set $\{2, 0, \infty\}$ is forward complete, but not backward complete. Moreover, the fact that, for instance, α^3 possess no outgoing edge means that there is no point in $C_2(\mathbb{F}_{25})$ above the point $\alpha^3 \in C_1(\mathbb{F}_{25})$; this also means that there is no points in $C_3(\mathbb{F}_{25})$ above the point $(\alpha^{16}, \alpha^3) \in C_2(\mathbb{F}_{25})$. In other terms, this point is inert in C_3 .

Lemma 13. *Let S be a finite and before complete subset of $X(\overline{\mathbb{F}_q})$ such that the graph \mathcal{G}_S is outside the singular part. Then S is complete.*

Proof — Since \mathcal{G}_S is outside the singular part and S is before complete, the in-degree at every vertex $P \in S$ is equal to $d^-(P) = d$. On the other side, the out-degree $d^+(P)$ at each vertex P is less than d . Counting the edges, we get

$$d\#S = \sum_{P \in S} d^-(P) = \sum_{P \in S} d^+(P) \leq d\#S,$$

so that one must have $d^+(P) = d$ for every $P \in S$, which means that S is also backward complete and thus complete. \square

Proposition 14. *Let (X, Γ) be a correspondence as in section 1.1. If there exists a finite complete set $S \subset X(\mathbb{F}_{q^r})$ such that the graph \mathcal{G}_S is outside the singular part \mathcal{G}_{sing} , then*

$$\#\tilde{C}_n(\mathbb{F}_{q^r}) \geq \#S \times d^{n-1}.$$

Proof — Since S is complete and since the graph \mathcal{G}_S is assumed to be outside the singular part, the graph \mathcal{G}_S must be d -regular. Each path of length n in \mathcal{G}_S gives rise to exactly d paths of length $n + 1$ by adding one of the d outgoing neighbors of the ending vertex. All these paths correspond to smooth points of C_n or C_{n+1} and we have just proved that above each such point of C_n , there is exactly d points on C_{n+1} . We easily conclude by induction since C_1 counts at least $\#S$ points defined over \mathbb{F}_{q^r} . \square

3.3 Graph aided study of the BGS tower

In 1985, Bezerra, Garcia and Stichtenoth [BGS05, BS07, BGS08] have introduced what we refer to as the BGS recursive tower $\mathcal{T}(X, \Gamma)$ over \mathbb{F}_q , defined by $X = \mathbb{P}^1$ and by the separated variables correspondence $\Gamma_{f,g}$ (notations of section 1.1) with

$$f(x) = \frac{x^q + x - 1}{x} \quad \text{and} \quad g(y) = \frac{1 - y}{y^q}.$$

This tower leads to the inequalities

$$A(q^3) \geq \lambda_3(X, \Gamma) \geq \frac{2(q^2 - 1)}{q + 2}.$$

This comes from the facts that the number of \mathbb{F}_{q^3} -rational points of \tilde{C}_n is proved to be equal to $q^n(q + 1) + o(q^n)$, and that the genus is proved to be at most $g_n \leq \frac{q+2}{2(q-1)} \times q^n$ (the JNT'07 article [BS07] is entirely devoted to a simplified proof of this last upper bound).

In this section, we show how the approach of this paper using singular tower, and how the graph point of view, permit to give a closed formula for the genus sequence g_n . We also give a bound for the number of geometric points coming from desingularization, useful for corollary 27.

Theorem 15. *Consider the BGS tower recursively defined by $\frac{x^q+x-1}{x} = \frac{1-y}{y^q}$.*

(i) *The geometric genus sequence is given by*

$$g_n = \frac{q^n}{2(q-1)} \left(q + 2 - \frac{3}{\lfloor \frac{n-1}{2} \rfloor} \right) \left(1 - \frac{1}{\lceil \frac{n-1}{2} \rceil} \right)$$

where $\lfloor x \rfloor$ denotes the greatest integer smaller than a real number x , and $\lceil x \rceil$ denotes the smallest integer greater than x .

(ii) *The number of geometric points of \tilde{C}_n coming from the desingularization of C_n is less than $3\sqrt{q^n}$.*

For each $n \geq 1$, the curve C_n is embedded in $(\mathbb{P}^1)^n = \prod_{i=1}^n \text{Proj}(\mathbb{F}_q[x_i, y_i])$ and is defined by the ideal

$$\langle x_{i+1}^q (x_i^q + x_i y_i^{q-1} - y_i^q) - (y_{i+1}^q - x_{i+1} y_{i+1}^{q-1}) x_i y_i^{q-1}, 1 \leq i \leq n-1 \rangle.$$

The totally split points — In figure 3.3, we represent two complete sets for $q = 3$. The left hand one counts $q(q + 1)$ vertices, all contained in \mathbb{F}_{q^3} . For $q = 5, 7, \dots$, one can easily see evidence of the existence of a complete set of size $q(q + 1)$ outside the singular part. By proposition 14, if this is true, one should have

$$\#\tilde{C}_n(\mathbb{F}_{q^3}) \geq q^n(q + 1).$$

Of course, this is not a proof of this fact, but only a convenient way to see it. This is proved in Crelle [BGS05, Proposition 3.1].

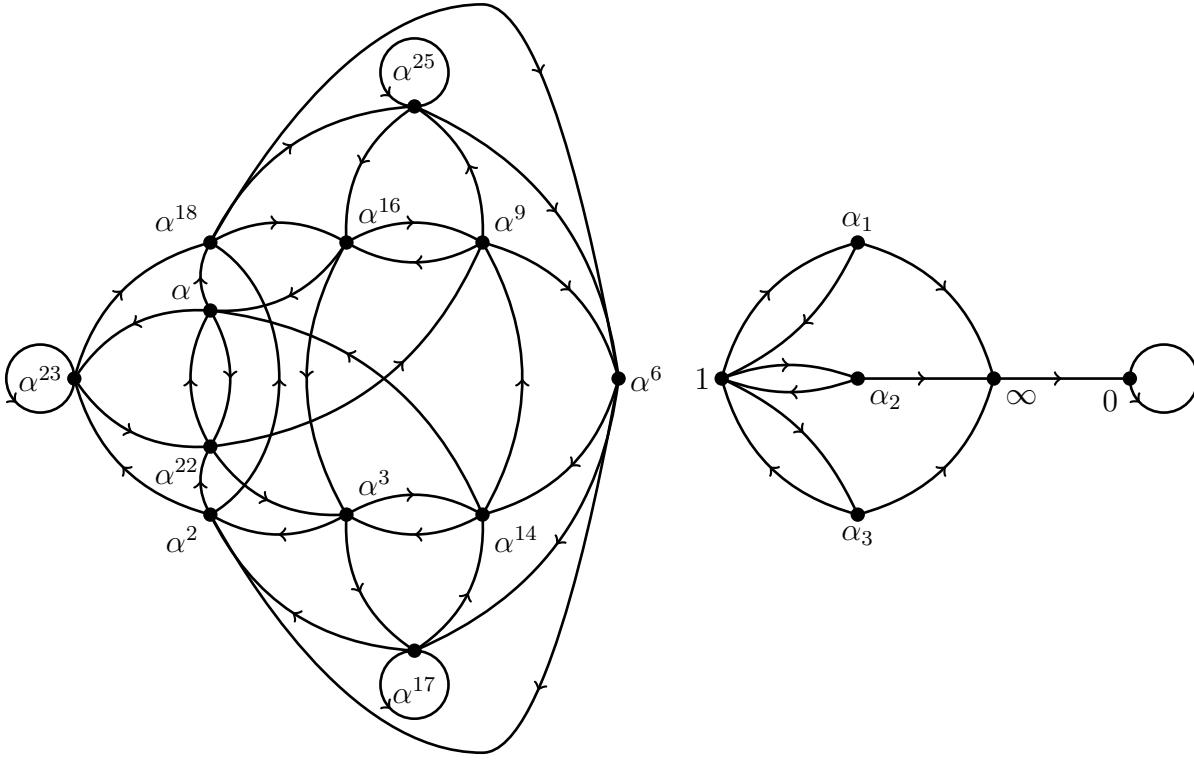


Figure 3: The two interesting components of \mathcal{G}_6 for $q = 3$

The singular points — The right hand side complete set of figure 3.3 has points $0, 1, \infty$ as vertices. These are exactly the ramified points of f or g : the ramified points of f (resp. g) are 1 and ∞ (resp. ∞ and 0). The set $\{0, 1, \infty\}$ is not complete but one can easily prove that it suffices to add the set \mathcal{R} of roots of $x^q + x - 1$ to complete the set. The subgraph $\mathcal{G}_{\{0,1,\infty\} \cup \mathcal{R}}$ is nothing else than the singular part $\mathcal{G}_{\text{sing}}$.

The fact that every curve C_n is irreducible can also be read of this component. Indeed, there is only one loop starting from the vertex 0 . This means that above the point $0 \in C_1$, there is only one point, i.e. $(0, \dots, 0) \in C_n$. Since 0 is not a ramified point of f , this point is smooth in C_n and then must be totally ramified over $0 \in C_1$. Necessarily C_n is irreducible⁶.

It can also be easily seen that for $\alpha_0, \dots, \alpha_r$ in \mathcal{R} , the points

$$(1, \alpha_1, 1, \alpha_2, \dots, 1, \alpha_r) \in C_{2r} \quad \text{and} \quad (\alpha_0, 1, \alpha_1, 1, \alpha_2, \dots, 1, \alpha_r) \in C_{2r+1}$$

are smooth. And to be a singular on C_n or $C_n^\#$, a point must start by 1 or ∞ (a ramified point of f) or by $\alpha \in \mathcal{R}$ (an incoming neighbor of a ramified point of f) and must end by 0 or ∞ (a ramified point of g). We distinguish two types of singular points depending on the ending point:

Type	Corresponding points on C_n	Range of r
T_∞	$(1, \alpha_1, 1, \alpha_2, \dots, 1, \alpha_r, \infty)$	n odd and $r = \frac{n-1}{2}$
	$(\alpha_0, 1, \alpha_1, 1, \alpha_2, \dots, 1, \alpha_r, \infty)$	n even and $r = \frac{n-2}{2}$
T_0	$(1, \alpha_1, 1, \alpha_2, \dots, 1, \alpha_r, \infty, 0, \dots, 0)$	$0 \leq r \leq \lfloor \frac{n-2}{2} \rfloor$
	$(\alpha_0, 1, \alpha_1, 1, \alpha_2, \dots, 1, \alpha_r, \infty, 0, \dots, 0)$	$0 \leq r \leq \lfloor \frac{n-3}{2} \rfloor$

In this tabular, the integer r is the number of instances of couples $(1, \alpha)$ for $\alpha \in \mathcal{R}$ in the considered point of C_n . If $r = 0$, there is no such couples; the two $r = 0$ case in type T_0 are points $(\infty, 0, \dots, 0)$ and $(\alpha_0, \infty, 0, \dots, 0)$. Along the proof of theorem 15, we prove that all these

⁶Using this argument, this is a general fact that if there exists in $\mathcal{G}_{\text{sing}}$ only one loop outgoing from a point, étale by π_2 , then the tower is irreducible. This is a common feature of many towers of the literature.

points give rise to a unique point on \widetilde{C}_n and on C_n^\sharp . We keep the same types to distinguish points on C_n^\sharp and \widetilde{C}_n .

In order to compute the geometric genus sequence in the spirit of section 2.1, we need three lemmas about the local situations above the singular points. These lemmas involve standard techniques, such as Newton polygons, in the area of explicit computation of integral closures. In fact, the lemmas describe specific steps related to our situation, in the `round4` algorithm due to Zassenhaus [Hal01] or in the algorithm developed by Montès and Nart [MN92].

Lemma 16. *Let A be a discrete valuation ring of characteristic p , K its field of fractions, k its residual field (assumed to be perfect), v the associated discrete valuation on K , and π an uniformizing element. Let x be a root of a separable unitary polynomial $F \in A[X]$ of degree n and such that $F \equiv X^n \pmod{\pi}$. Suppose that the Newton polygon of F consists in a unique segment of slope $-\frac{\nu}{n}$ with $\gcd(n, \nu) = 1$. Then the free A -module*

$$B = \bigoplus_{j=0}^{n-1} A \frac{x^j}{\pi^{\lfloor \frac{\nu j}{n} \rfloor}}$$

is the integral closure of $A[x]$ in $K[x]$ and $\delta_{B/A[x]} = \dim_k(B/A[x]) = \frac{(n-1)(\nu-1)}{2}$.

Proof — For $j \in \mathbb{N}$, put $x_j = \frac{x^j}{\pi^{\lfloor \frac{\nu j}{n} \rfloor}}$ and $q_j = \lfloor \frac{\nu j}{n} \rfloor$. The fact that the Newton polygon contains only one segment implies that every valuation w of $K[x]$ extending v satisfies $w(x) = \frac{\nu}{n}$. Therefore, we have $w(x_j) = \frac{j\nu}{n} - \lfloor \frac{\nu j}{n} \rfloor \geq 0$ and x_j is integral over A . The module B is thus included in the integral closure.

To prove that B is equal to this closure, we first verify that B is a ring. Dividing the relation $F(x) = 0$ by π^ν and taking into account the fact that the polygon has only one segment, we prove that $x_n = \frac{x^n}{\pi^\nu} \in B$. For $j, r \in [0..n-1]$, we write $j+r = \epsilon n + s$ with $\epsilon \in \{0, 1\}$ and we remark that

$$x_j \times x_r = \pi^{q_{j+r} - (q_j + q_r)} x_s \times x_n^\epsilon,$$

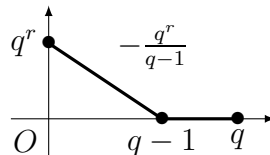
with $q_{j+r} - (q_j + q_r) \geq 0$ and $s < r$ if $\epsilon = 1$. By induction on r , this permits to show that $x_i \times x_j \in B$ for every $0 \leq i, j < n$, and that B is a ring. Then we note that v must be totally ramified in $K[x]$ and that B contains an uniformizing element. Indeed $x_{\nu^{-1} \bmod n}$ is in B and satisfies $w(x_{\nu^{-1} \bmod n}) = \frac{1}{n}$.

Finally $\delta_{B/A[x]} = \sum_{j=0}^{n-1} \lfloor \frac{\nu j}{n} \rfloor$, which is known to be equal to $\frac{\gcd(\nu-1, n-1)}{2}$. \square

Lemma 17. *Let A be a discrete valuation ring of characteristic p , K its field of fractions, v the associated discrete valuation on K and π an uniformizing element. Let x be a root of the polynomial $X^q - X^{q-1} - \mu\pi^{q^r}$ where μ is an invertible element of A . Then*

- (i) *the valuation v factorizes into $w^{q-1}w'$ where w, w' are valuations of $K[x]$, w being ramified over K with index $(q-1)$, and w' being unramified over K ;*
- (ii) *these (normalized) valuations w, w' of $K[x]$ extending v are such that $w(x) = q^r$ and $w'(x) = 0$;*
- (iii) *the integral closure B of $A[x]$ in $K[x]$ satisfies: $\delta_{B/A[x]} = \frac{(q-2)(q^r-1)}{2}$.*

Proof — The Newton polygon has shape

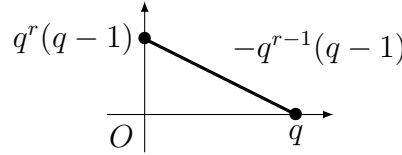


We deduce that there exist two (non normalized) valuations w and w' of $K[x]$ extending v such that $w(x) = \frac{q^r}{(q-1)}$ and $w'(x) = 0$. Because of the denominator $(q-1)$, the valuation w must be ramified over K of index $(q-1)$. This proves, point (i). Point (ii) is just a matter of normalization. Point (iii) is a consequence of lemma 16 applied to the π -adic factor corresponding to the segment of slope $\frac{q^r}{(q-1)}$. \square

Lemma 18. *Let A be a discrete valuation ring of characteristic p , K its field of fractions, v the associated discrete valuation of K and π an uniformizing element. Let x be a root of the polynomial $X^q + \lambda\pi^{q^r(q-1)}X + \mu\pi^{q^r(q-1)}$ where λ, μ are invertible elements of A . Then:*

- (i) *the valuation v is totally ramified in $K[x]$;*
- (ii) *the unique (normalized) valuation w of $K[x]$ extending v is such that $w(x) = q^r(q-1)$;*
- (iii) *the integral closure B of $A[x]$ in $K[x]$ satisfies $\delta_{B/A[x]} = \frac{(q-1)(q^{r+1}-2)}{2}$.*

Proof — The Newton polygon has shape



We deduce that every (non normalized) valuation \tilde{v} of $K[x]$ extending v is such that $\tilde{v}(x) = q^{r-1}(q-1)$.

We distinguish two cases:

- The case $r = 0$ is a direct consequence of lemma 16, since the Newton polygon has a unique segment of slope $-\frac{q-1}{q}$. Note that, after normalization, the unique valuation w of $K[x]$ extending v satisfies $w(x) = q-1$.
- If $r > 0$, then $z_1 = \frac{x}{\pi^{q^{r-1}(q-1)}}$ is an integer and satisfies

$$X^q + \lambda\pi^{q^{r-1}(q-1)}X + \mu = (X + \mu^{1/p})^p + \lambda\pi^{q^{r-1}(q-1)}(X + \mu^{1/p}) - \lambda\mu^{1/p}\pi^{q^{r-1}(q-1)}$$

$$\delta_{A[z_1]/A[x]} \sim \pi^{q^{r-1}(q-1) \sum_{i=0}^{q-1} i} \sim \pi^{\frac{q^r(q-1)^2}{2}}.$$

By iterating this process, we compute a sequence of integers $z_0 = x, z_1, \dots, z_r$ such that

$$A[x] = A[z_0] \subset A[z_1] \subset \dots \subset A[z_r], \quad \forall 1 \leq i \leq r, \quad \delta_{A[z_i]/A[z_{i-1}]} = \frac{q^i(q-1)^2}{2}.$$

Moreover the last integer z_r satisfies an equation like in case $r = 0$.

In conclusion

$$\delta_{B/A[x]} = \frac{(q-2)(q-1)}{2} + \frac{(q-1)^2}{2} \sum_{i=1}^r q^i = \frac{(q-2)(q-1)}{2} + \frac{(q-1)q(q^r-1)}{2}$$

and the result follows. \square

We are now able to compute the different measure of singularity and then to compute the geometric genus sequence of the tower.

Lemma 19. *The two kinds of types give rise to the following singularities.*

(i) The points of type T_∞ have a measure of singularity equal to

$$\delta_\infty(n) = \frac{(q^{\lfloor \frac{n-1}{2} \rfloor} - 1)(q - 2)}{2}.$$

(ii) The points of type T_0 have a measure of singularity which only depends on r

$$\delta_0(r) = \frac{(q^{r+1} - 2)(q - 1)}{2}.$$

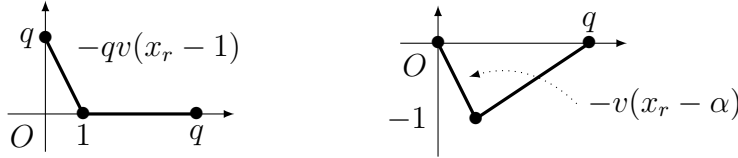
(iii) This leads to

$$\Delta_n = q^n \left(1 - \frac{1}{q}\right) + \frac{q+2}{2} - \frac{1}{2}q^{n-\lfloor \frac{n-1}{2} \rfloor} - q^{n-\lceil \frac{n-1}{2} \rceil}.$$

Proof — Let $\alpha \in \mathcal{R}$ be a root of $x^q + x - 1$. The polynomial relying x_r and x_{r+1} can be written like this

$$(x_{r+1} - \alpha)^q + \frac{x_r}{x_r^q + x_r - 1} (x_{r+1} - \alpha) + \frac{(1 - \alpha)(x_r - 1)^p}{x_r^q + x_r - 1} \quad \text{and} \quad (x_{r+1} - 1)^q + \frac{x_r}{x_r^q + x_r - 1} (x_{r+1} - 1) + 1.$$

The $(x_r - 1)$ and $(x_r - \alpha)$ corresponding polygons are:



Therefore above any valuation v of $\mathbb{F}_q(C_r)$ such that $v(x_r - 1) > 0$ (resp. $v(x_r - \alpha) > 0$), there is a unique valuation w of $\mathbb{F}_q(C_{r+1})$ such that $w(x_{r+1} - \alpha) > 0$ (resp. $w(x_{r+1} - 1) > 0$); more precisely, one has $w(x_{r+1} - \alpha) = qv(x_r - 1)$ (resp. $w(x_{r+1} - 1) = v(x_r - \alpha)$). By an easy induction, we deduce that for any $\alpha_0, \alpha_1, \dots, \alpha_r \in \mathcal{R}$, the unique (normalized) valuation v of $\mathbb{F}_q(C_{2r})$ or $\mathbb{F}_q(C_{2r+1})$ corresponding to the smooth point $(1, \alpha_1, \dots, 1, \alpha_r) \in C_{2r}$ or $(\alpha_0, 1, \alpha_1, \dots, 1, \alpha_r) \in C_{2r+1}$ satisfy

$$v(x_{2r} - \alpha_r) = q^r \quad \text{or} \quad v(x_{2r+1} - \alpha_r) = q^r.$$

• Let P be the smooth point on C_{n-1} defined by

$$P = ([\alpha_0], 1, \alpha_1, \dots, 1, \alpha_{\lfloor \frac{n-1}{2} \rfloor}) \in C_{n-1},$$

where the brackets around α_0 means that it could appear or not depending on the parity of n . Let v denotes the corresponding valuation of $\mathbb{F}_q(C_{n-1})$. We consider the singular point $Q = (P, \infty)$ of type T_∞ on C_n^\sharp . The relation between the functions x_{n-1} and y_n is

$$y_n^q - y_n^{q-1} - \frac{x_{n-1}^q + x_{n-1} - 1}{x_{n-1}} = 0.$$

In view of the beginning of this proof, we know that

$$v\left(\frac{x_{n-1}^q + x_{n-1} - 1}{x_{n-1}}\right) = q^{\lfloor \frac{n-2}{2} \rfloor}.$$

Due to lemma 17, we know that there is only one point $\tilde{Q} \in \tilde{C}_n$ above Q . Moreover, the corresponding valuation w of $\mathbb{F}_q(C_n)$ is such that $w(y_n) = q^{\lfloor \frac{n-1}{2} \rfloor}$ and the measure of singularity is $\delta_\infty(n)$ as in (i).

• Let P denote the unique point on \tilde{C}_{n-1} above $([\alpha_0], 1, \alpha_1, \dots, 1, \alpha_{\lfloor \frac{n-2}{2} \rfloor}, \infty) \in C_{n-1}$ (note that if $n = 2$, this includes the point ∞ on C_1) and let v be the corresponding (normalized) valuation of $\mathbb{F}_q(C_{n-1})$. We consider the singular point $Q = (P, 0) \in C_n^\sharp$. The relation between x_n and y_{n-1} is

$$x_n^q + \frac{y_{n-1}^{q-1}}{1 + y_{n-1}^{q-1} - y_{n-1}^q} x_n - \frac{y_{n-1}^{q-1}}{1 + y_{n-1}^{q-1} - y_{n-1}^q} = 0,$$

and thanks to the preceding point we know that

$$v \left(\frac{y_{n-1}^{q-1}}{1 + y_{n-1}^{q-1} - y_{n-1}^q} \right) = (q-1)q^{\lfloor \frac{n-1}{2} \rfloor}.$$

Using lemma 18, we deduce that there is a unique point on \tilde{C}_n above Q , that the corresponding valuation w of $\mathbb{F}_q(C_n)$ is such that $w(x_n) = (q-1)q^{\lfloor \frac{n-2}{2} \rfloor}$ and that the measure of singularity is equal to $\delta_0(\lfloor \frac{n-2}{2} \rfloor)$.

• Let P denote the unique point on \tilde{C}_{n-1} above $([\alpha_0], 1, \alpha_1, \dots, 1, \alpha_r, \infty, 0, \dots, 0) \in C_{n-1}$ and let v be the corresponding (normalized) valuation on $\mathbb{F}_q(C_{n-1})$. One should start with only one zero but the proof does not change with more zeros. We consider the singular point $Q = (P, 0) \in C_n^\sharp$. The relation between x_n and x_{n-1} is

$$x_n^q + \frac{x_{n-1}}{x_{n-1}^q + x_{n-1} - 1} x_n - \frac{x_{n-1}}{x_{n-1}^q + x_{n-1} - 1} x_n = 0,$$

and thanks to the preceding point we know that

$$v \left(\frac{x_{n-1}}{x_{n-1}^q + x_{n-1} - 1} \right) = v(x_{n-1}) = (q-1)q^r.$$

Using lemma 18, we deduce that there is a unique point on \tilde{C}_n above Q , that the corresponding valuation w of $\mathbb{F}_q(C_n)$ is such that $w(x_n) = (q-1)q^r$ and that the measure of singularity is equal to $\delta_0(r)$.

• To compute Δ_n , we only have to sum all the measures of singularity of all the singular points. We separate this sum in three terms, one for the singular points ending by ∞ , one for the singular points ending by zero and starting by 1 and one for the singular points ending by zero and starting by $\alpha \in \mathcal{R}$

$$\Delta_n = q^{\lfloor \frac{n-1}{2} \rfloor + (n-1) \bmod 2} \delta_\infty(n) + \sum_{r=0}^{\lfloor \frac{n-2}{2} \rfloor} q^r \delta_0(r) + \sum_{r=1}^{\lfloor \frac{n-3}{2} \rfloor} q^{r+1} \delta_0(r).$$

It is then just a matter involving only summations of geometric series to deduce assertion (iii) from (i) and (ii). For instance if $n = 2m + 1$ is odd, then

$$\begin{aligned} \Delta_{2m+1} &= q^m \frac{(q^m - 1)(q - 2)}{2} + \sum_{r=0}^{m-1} q^r \frac{(q^{r+1} - 2)(q - 1)}{2} + \sum_{r=0}^{m-1} q^{r+1} \frac{(q^{r+1} - 2)(q - 1)}{2} \\ &= q^m \frac{(q^m - 1)(q - 2)}{2} + \frac{q - 1}{2} \times \left(q \sum_{r=0}^{m-1} q^{2r} - 2 \sum_{r=0}^{m-1} q^r + q^2 \sum_{r=0}^{m-1} q^{2r} - 2q \sum_{r=0}^{m-1} q^r \right) \\ &= \frac{q^{2m+1} - 2q^{2m} - q^{m+1} + 2q^m}{2} + \frac{q - 1}{2} \times \left(q(q + 1) \frac{q^{2m} - 1}{q^2 - 1} - 2(q + 1) \frac{q^m - 1}{q - 1} \right) \\ &= \frac{2q^{2m+1} - 2q^{2m} - 3q^{m+1} + q + 2}{2}. \end{aligned}$$

The even case works in the same way. □

Proof of theorem 15 — The computation of the genus follows by summations of geometric series from item (iii) of lemma 19 and item (ii) of lemma 4, taking into account that by adjunction formula, the arithmetic genus γ_2 of Γ satisfies $\gamma_2 - 1 = q(q - 2)$. For instance, if $n = 2m + 1$ is odd, then

$$\begin{aligned}
g_{2m+1} &= 1 + (n - 1)q^{n-1}[q(q - 2) + q] - q^{n-1} - \sum_{i=2}^n q^{n-i} \Delta_i \\
&= 1 + (n - 1)q^n \left(1 - \frac{1}{q}\right) - q^{n-1} - (n - 1)q^n \left(1 - \frac{1}{q}\right) - \frac{q + 2}{2} \frac{q^{2m} - 1}{q - 1} \\
&\quad + \frac{q^n}{2} \sum_{i=2}^n \frac{1}{q^{\lfloor \frac{i-1}{2} \rfloor}} + q^n \sum_{i=2}^n \frac{1}{q^{\lceil \frac{i-1}{2} \rceil}} \\
&= 1 - q^{n-1} - \frac{q + 2}{2(q - 1)}(q^{2m} - 1) + \frac{q^n}{2} \left(1 + \frac{1}{q}\right) \frac{1 - \frac{1}{q^m}}{1 - \frac{1}{q}} + 2q^{n-1} \frac{1 - \frac{1}{q^m}}{1 - \frac{1}{q}} \\
&= -q^{2m} \left(1 - \frac{1}{q^m}\right) \left(1 + \frac{1}{q^m}\right) - \frac{q^n}{2(q - 1)} \frac{q + 2}{q} \left(1 - \frac{1}{q^m}\right) \left(1 + \frac{1}{q^m}\right) \\
&\quad + \frac{q^n}{2(q - 1)} \left(1 - \frac{1}{q^m}\right) (q - 1) + \frac{2q^n}{q - 1} \left(1 - \frac{1}{q^m}\right) \\
&= \frac{q^n}{2(q - 1)} \left(1 - \frac{1}{q^m}\right) \left(-2 \left(1 - \frac{1}{q}\right) \left(1 + \frac{1}{q^m}\right) - \left(1 + \frac{2}{q}\right) \left(1 + \frac{1}{q^m}\right) + q + 1 + 4\right) \\
&= \frac{q^n}{2(q - 1)} \left(1 - \frac{1}{q^m}\right) \left(q + 2 - \frac{3}{q^m}\right),
\end{aligned}$$

and the calculation is similar if $n = 2m$. □

4 Application to the asymptotic behavior of recursive towers

In this section, we apply the preceding results, and we use spectral graph theory and intersection theory on $X \times X$. We prove theorem 23 that under the assumptions of section 1.1, there exists at most one finite strongly connected component in the geometric graph \mathcal{G}_∞ . We deduce from this theorem 24, that at most one β_r is non zero. As a corollary, we deduce some invariants defined by Tsfasmann and Vladut in [TV02] for the BGS tower already studied in section 3.3. The key point for this section is proposition 20, an inequality blending combinatoric and geometry. A diophantine lemma 21 is very useful to use this proposition 20.

4.1 Number of cycles

For the statement of the next proposition, we recall that the numerical class in the Neron-Severi group $\text{NS}(X \times X)_\mathbb{R}$ of a correspondence C in the surface $X \times X$ is a triple $(d_1, d_2, \sigma) \in \mathbb{Z} \times \mathbb{Z} \times \text{End}(T_\ell(\text{Jac}(X)))$ where $\text{Jac}(X)$ is the Jacobian variety of X , and $T_\ell(\text{Jac}(X))$ is its Tate module for some prime number ℓ prime to q . For instance, the class of the diagonal Δ is $(1, 1, \text{Id})$. Then, the intersection number $C \cdot C'$ is given by

$$C \cdot C' = (d_1, d_2, \sigma) \cdot (d'_1, d'_2, \sigma') = d_1 d'_2 + d'_1 d_2 - \text{tr}(\sigma \sigma'). \quad (16)$$

Moreover, Castelnuovo identity states that the bilinear form $\text{tr}(\sigma \sigma')$ is negative definite ([Zar95] chapter VII, appendix of Mumford p. 153).

It worth to notice that what we called up to now the type (d_1, d_2) of a divisor C in $X \times X$ is actually the “trivial” part of its complete numerical class (d_1, d_2, σ) .

The nice feature in the following statement is that the formula (17) has a left hand side of combinatoric nature, whereas its right hand side is of geometric nature.

Proposition 20. *Let (X, Γ) be a correspondence as in section 1.1, let $(C_n)_{n \geq 1}$ be the associated recursive tower, let (d, d, σ) be the numerical class of Γ in $\text{NS}(X \times X)_{\mathbb{R}}$, and let c_n denote the number of cycles of length n in \mathcal{G}_{∞} . We assume that C_n is irreducible for any $n \geq 1$. Then c_n is finite and for every $n \geq 1$ and for every $r \geq 1$ such that the graph \mathcal{G}_r contains the cycles of length n , one has*

$$c_n = \sum_{\lambda \in \text{Sp}(A_r)} \lambda^n \leq 2d^n - \sum_{\lambda \in \text{Sp}(\sigma)} \lambda^n, \quad (17)$$

where A_r is the adjacency matrix of \mathcal{G}_r . Moreover the spectral radius of σ is at most equal to d .

Proof — Let $\pi_{1,n+1}$ denotes the projection map $X^{n+1} \rightarrow X \times X$ which sends (P_1, \dots, P_{n+1}) to (P_1, P_{n+1}) and Δ denotes the diagonal of $X \times X$. A geometric point $(P_1, \dots, P_{n+1}) \in X^{n+1}$ corresponds to a cycle of length n in \mathcal{G}_{∞} if and only if $(P_1, \dots, P_{n+1}) \in C_{n+1}$ and $(P_1, \dots, P_{n+1}) \in \pi_{1,n+1}^*(\Delta)$, which means that cycles of length n corresponds to points in the intersection cycle $C_{n+1} \cdot \pi_{1,n+1}^*(\Delta)$ in X^{n+1} . By projection formula, its degree equals that of intersection cycle $(\pi_{1,n+1})_*(C_{n+1}) \cdot \Delta$ in the surface $X \times X$.

We begin by proving that the irreducible curve C_{n+1} is not contained in the hypersurface $\pi_{1,n+1}^*(\Delta)$. From the hypothesis on Γ , the first projection $\pi_1 : \Gamma \rightarrow X$ is a finite morphism of degree d , étale except at a finite number of geometric points $(P, Q) \in \Gamma$. Choose a geometric point $P \in X$, such that π_1 is étale at any point $(P, Q_i) \in \Gamma$, $1 \leq i \leq d$ lying above P . Choose a geometric point $(P_1, \dots, P_{n-2}, P) \in C_{n-1}$ whose last coordinate is $P_{n-1} = P$. There are d distinct geometric points $(P_1, \dots, P_{n-2}, P, Q_i) \in C_n$, for $1 \leq i \leq d$ lying above $(P_1, \dots, P_{n-2}, P) \in C_{n-1}$. Suppose now by contradiction that $C_n \subset \pi_{1,n+1}^*(\Delta)$. This means that for any $1 \leq i \leq d$, we have $Q_i = P_1$, a contradiction since $d \geq 2$.

It follows that the intersection $C_{n+1} \cap \pi_{1,n+1}^*(\Delta)$ in X^{n+1} is a zero dimensional subvariety whose geometric points correspond to the cycles of length n . This proves that c_n is finite. Taking into account multiplicities, we deduce that

$$\begin{aligned} c_n &\leq C_{n+1} \cdot \pi_{1,n+1}^*(\Delta) && \text{computed in } X^{n+1} \\ &= (\pi_{1,n+1})_*(C_{n+1}) \cdot \Delta \in X^2 && \text{computed in } X^2 \end{aligned}$$

by projection formula. But the numerical class of $(\pi_{1,n+1})_*(C_{n+1})$ is (d^n, d^n, σ^n) and the diagonal's one equals $(1, 1, \text{Id})$, hence by (16)

$$C_{n+1} \cdot \pi_{1,n+1}^*(\Delta) = (\pi_{1,n+1})_*(C_{n+1}) \cdot \Delta = d^n \times 1 + d^n \times 1 - \text{tr}(\sigma^n).$$

Formula (17) follows. Since the number of cycles of length n is well known to be the trace of the n -th power of the adjacency matrix A_r of \mathcal{G}_r .

Finally, since this intersection is an effective zero cycle on $X \times X$, its degree is a positive number, that is

$$\sum_{\lambda \in \text{Sp}(\sigma)} \lambda^n \leq 2d^n$$

for any $n \geq 1$. It follows that $|\lambda| \leq d$ for any $\lambda \in \text{Sp}(\sigma)$. \square

Remark — In case $X = \mathbb{P}^1$, it is also possible to give a proof of the first assertion of this proposition using resultants, while the second assertion is empty.

This proposition is fruitful in conjunction with the following lemma:

Lemma 21 (Diophantine approximation). *Let $\lambda_1, \dots, \lambda_k \in \mathbb{C}^*$. Then there exists an integer $N \in \mathbb{N}^*$, such that $\Re(\lambda_j^N) > 0$ for each $1 \leq j \leq k$.*

Proof — Let $\mu_j = \frac{\lambda_j}{|\lambda_j|}$ for $1 \leq j \leq k$. Then $\mu_j = \exp(2i\pi\theta_j)$ for some real number $\theta_j \in \mathbb{R}$. It follows from Hardy and Wright [HW08, theorem 201] that for any $\varepsilon > 0$, there exists some $N \in \mathbb{N}^*$ such that $d(N\theta_j, \mathbb{Z}) < \varepsilon$ for all $1 \leq j \leq k$. By continuity of the exponential map, we can also choose N such that $|\mu_j^N - 1| < \varepsilon$ for any $1 \leq j \leq k$. Now we choose $\varepsilon = 1$. There exists some $N \in \mathbb{N}$, such that $|\lambda_j^N - |\lambda_j^N|| < |\lambda_j^N|$ for any j , which implies that $\Re(\lambda_j^N) > 0$. \square

4.2 Finite strongly connected regular components

We refer to Godsil & Royle's GTM [GR01] for results about graph theory. We should focus on *strongly connected components* of the graph \mathcal{G}_∞ and especially the finite ones. Recall that a directed graph is said to be *strongly connected* if there is a path from each vertex to every other vertex. A *strongly connected component* of a graph is a maximal strongly connected subgraph. Such a component is said to be *primitive* if there is a path of *common length* between every couple of vertices (see loc. cit. §2.6).

In terms on adjacency matrix, a finite graph is strongly connected (resp. primitive) if and only if its adjacency matrix is irreducible (resp. primitive). The Perron-Frobenius theorem (see loc. cit. Theorem 8.8.1) deals with the spectrum of such adjacency matrices.

For a finite *regular* directed graph, being weakly connected is equivalent to being strongly connected (see loc. cit. Lemma 2.6.1). As a consequence, every strongly connected component of a finite *regular* directed graph is still regular.

Proposition 22. *Let (X, Γ) be a correspondence as in section 1.1 and let (d, d, σ) be the class of Γ in $\text{NS}(X \times X)_{\mathbb{R}}$. Then every finite d -regular strongly connected subgraph \mathcal{G} of the graph $\mathcal{G}_\infty(X, \Gamma)$ is primitive.*

Proof — Let A be the adjacency matrix of the subgraph \mathcal{G} . Since \mathcal{G} is supposed to be strongly connected, the matrix A is irreducible. Since \mathcal{G} is d -regular, the vector $(1, \dots, 1)$ is an eigenvector of A for the eigenvalue d . By Perron-Frobenius theorem, this eigenvalue is simple and is nothing else than the spectral radius of A . Moreover there exists a primitive root of unity ζ_m such that the eigenvalues of absolute value d are the $d\zeta_m^i$ for $0 \leq i \leq m-1$, and all these eigenvalues are also simple.

Relating to the trace of the matrix A^{mn} for $n \geq 1$, this implies that

$$\text{tr}(A^{mn}) = md^{mn} + \sum_{\substack{\lambda \in \text{Sp}(A) \\ |\lambda| < d}} \lambda^{mn}$$

But this trace is also the number of cycles of length mn in \mathcal{G} . By proposition 20, we thus have

$$\forall n \geq 1, \quad (m-2)d^{mn} + \sum_{\substack{\lambda \in \text{Sp}(A) \\ |\lambda| < d}} \lambda^{mn} + \sum_{\lambda \in \text{Sp}(\sigma)} \lambda^{mn} \leq 0.$$

Note that in the left sum, any $\lambda \in \mathbb{C} \setminus \mathbb{R}$ appears together with its conjugate $\bar{\lambda}$. Then this sum is, in fact, a sum of real part of powers of complex numbers. By lemma 21, we deduce that $m \leq 2$. Moreover, if $m = 2$, all the eigenvalues λ 's in the left sum must be equal to zero using lemma 21 another time. Then the number of cycle of length $2n$ in \mathcal{G} , counted without multiplicities, is

exactly $2d^{2n}$. But by proposition 20, it is also equal to $2d^{2n}$ counted with multiplicities. Then, for all $n \geq 1$, every cycle of length $2n$ must be simple. Thanks to corollary 9, we know that this is impossible. Hence $m = 1$; this characterizes the fact that the matrix A or the graph \mathcal{G} are primitive. \square

Theorem 23. *Let (X, Γ) be a correspondence as in section 1.1 such that the curves C_n of the associated tower are all irreducible. Then the graph $\mathcal{G}_\infty(X, \Gamma)$ has at most one finite d -regular strongly connected component.*

Remark – This theorem contains as a particular case Beelen’s theorem 5.5 in the case of towers of type A on $X = \mathbb{P}^1$.

Proof — Suppose that there exists at least one such component. Let $\mathcal{G}_1, \dots, \mathcal{G}_k$, some finite d -regular strongly connected components of \mathcal{G}_∞ and let A_i , $1 \leq i \leq k$, be their adjacency matrices. We denote by $\text{Sp}(A_i)$ the spectrum of each A_i . As noticed in the preceding proof, each matrix A_i has spectral radius d , and d is a simple eigenvalue. Hence, for any $n \geq 1$

$$\text{tr}(A_1^n) + \dots + \text{tr}(A_k^n) = kd^n + \sum_{\lambda \in \bigcup_{i=1}^k \text{Sp}(A_i) \setminus \{d\}} \lambda^n.$$

But this sum of traces is also the number of cycles of length n in the union of the \mathcal{G}_i ’s for $1 \leq i \leq k$, which is of course less than the number of cycles of length n in the arithmetic graph \mathcal{G}_r for r large enough. Now, we have assumed that there exists at least one finite d -regular strongly connected component, which contains of course at least one cycle of some length. Taken sufficiently often, this cycle has multiplicity at least 2 by corollary 9, that is there is at least one cycle, of some length $m \in \mathbb{N}^*$, having multiplicity at least 2. Because the number of cycles of some length n , counted with multiplicities is known to be less than $2d^n - \sum_{\lambda \in \text{Sp}(\sigma)} \lambda^n$ by proposition 20, we have for any $n \geq 1$,

$$kd^{mn} + \sum_{\lambda \in \bigcup_{i=1}^k \text{Sp}(A_i) \setminus \{d\}} \lambda^{mn} + \sum_{\lambda \in \text{Sp}(\sigma)} \lambda^{mn} \leq 2d^{mn} - 1$$

that is

$$\sum_{\lambda \in \text{Sp}(\sigma) \cup (\text{Sp}(A_1) \setminus \{d\}) \cup \dots \cup (\text{Sp}(A_k) \setminus \{d\})} (\lambda^m)^n \leq (2 - k)d^{mn} - 1.$$

Due to lemma 21, there exists some $N \in \mathbb{N}^*$ such that

$$0 \leq (2 - k)d^{mN} - 1,$$

which implies that $k \leq 1$. \square

4.3 Exactness of recursive towers

Theorem 24. *Let (X, Γ) be a correspondence as in section 1.1. Suppose that the curves C_n of the associated tower are all irreducible and that the geometric genus sequence $(g_n)_{n \geq 1}$ goes to $+\infty$. Suppose also that:*

- (i) *the singular points of C_n give rise to a number of geometric points in \tilde{C}_n negligible in front of d^n for large n .*
- (ii) *the graph \mathcal{G}_∞ contains at least one (hence exactly one by theorem 23) finite d -regular strongly connected component.*

Then, there exists at most one integer $r \geq 1$ such that $\beta_r \neq 0$.

Remarks

1. Up to our knowledge, the only example in the literature which does not satisfy (i) is the tower defined by the recursive equation $(x+1)^3 = y^3 + 1$ over \mathbb{F}_4 ([GST97] or [Bee04, example 2.4]).
2. This is false for good towers constructed from Hilbert class field towers using Grunwald-Wang theorem as communicated to us by Philippe Lebaque.

The proof of theorem 24 involves the following lemmas, for which graph theory is again the key tool. For any $S \in X(\overline{\mathbb{F}}_q)$, let A_S be the adjacency of the graph \mathcal{G}_S defined in definition 10.

Lemma 25. *Let (X, Γ) be a correspondence as in section 1.1 and let $S \subset X(\overline{\mathbb{F}}_q)$ be a finite subset. If d is an eigenvalue of A_S , then \mathcal{G}_S contains a d -regular strongly connected component outside the singular part $\mathcal{G}_{\text{sing}}$.*

Proof of lemma 25 — Since A_S is a nonnegative matrix and the sums on any line and any column of A_S are between 0 and d , its spectral radius $\rho(A_S)$ satisfies

$$0 \leq \rho(A_S) \leq d.$$

Since moreover d is assumed to be an eigenvalue of A_S , we have $\rho(A_S) = d$. By theorem 8.3.1 of Horn & Johnson's book about matrix analysis⁷ [HJ90] we know that d is associated to a nonnegative eigenvector

$$u = (u_P)_{P \in S}, \text{ such that } u_P \geq 0 \text{ for any } P \in S.$$

Of course, u_P may be zero for some $P \in S$. Let $\Sigma \subset S$ be the set of $P \in S$ such that $u_P \neq 0$.

Let $A_S = (a_{P,Q})_{P,Q \in S}$ and $A_\Sigma = (a_{P,Q})_{P,Q \in \Sigma}$. Then

$$\forall P \in \Sigma, \quad \sum_{Q \in S} a_{P,Q} u_Q = \sum_{Q \in \Sigma} a_{P,Q} u_Q = d u_P,$$

which means that the positive vector $(u_P)_{P \in \Sigma}$ is an eigenvector of A_Σ for the eigenvalue d . The matrix A_Σ is nothing else than the adjacency matrix of the subgraph \mathcal{G}_Σ .

Now, we prove that \mathcal{G}_Σ is d -regular, outside the singular part. By summation

$$\sum_{Q \in \Sigma} \left(\sum_{P \in \Sigma} a_{P,Q} \right) u_Q = \sum_{Q \in \Sigma} d u_Q.$$

But:

- each u_Q is > 0 for $Q \in \Sigma$,
- each $(\sum_{P \in \Sigma} a_{P,Q})$ satisfies $\sum_{P \in \Sigma} a_{P,Q} \leq d$.

Hence

$$\forall Q \in \Sigma, \quad \sum_{P \in \Sigma} a_{P,Q} = d.$$

Let $Q \in \Sigma$; each term in the Q 'th column in the adjacency matrix A_Σ contains exactly d coefficients 1. This means first that π_1 is étale at any edge exiting from Q , second that Σ is forward complete. Using a similar argument with the lines of A_Σ , we also prove that π_2 is étale at any edge entering at Q , and that Σ is also backward complete, hence complete. In conclusion, the graph \mathcal{G}_Σ is d regular and any of its strongly connected components works. \square

⁷Let us note that this result is again a consequence of the Perron-Frobenius theorem.

For any $r \geq 1$, we denote by $\mathcal{G}_r^{\text{smooth}} = \mathcal{G}_r \setminus \mathcal{G}_{\text{sing}}$, by $S_r^{\text{smooth}} \subset X(\mathbb{F}_{q^r})$ its support and by A_r^{smooth} its adjacency matrix. The spectral radius ρ_r^{smooth} of A_r^{smooth} is less than d , and by lemma 25, we have

$$\rho_r^{\text{smooth}} = d \iff \mathcal{G}_r^{\text{smooth}} \text{ contains a } d\text{-regular strongly connected component } \mathcal{G}_{\Sigma_r}.$$

Moreover, in this case the strongly connected component Σ_r is unique by theorem 23. We have:

Lemma 26. *With the notations above,*

(i) *if $\rho_r^{\text{smooth}} = d$, then $||| (A_r^{\text{smooth}})^n |||_1 = \#(\Sigma_r) \times d^n + o(d^n)$;*

(ii) *if $\rho_r^{\text{smooth}} < d$, then $||| (A_r^{\text{smooth}})^n |||_1 = o(d^n)$.*

Proof — Suppose that $\rho_r^{\text{smooth}} = d$. Let B be the adjacency matrix of the d -regular strongly connected component \mathcal{G}_{Σ_r} and C be the adjacency matrix of its complement in $\mathcal{G}_r^{\text{smooth}}$. Then A_r^{smooth} is a 2×2 block-diagonal matrix whose diagonal entries are B and C . The norm of B^n equals the number of paths of length n in \mathcal{G}_{Σ_r} , that is equals $\#(\Sigma_r) \times d^n$ by d -regularity. Moreover, the spectral radius $\rho(C)$ of C is $< d$ by lemma 25 and theorem 23. Since $\rho(C) = \lim_{n \rightarrow \infty} \sqrt[n]{|||C^n|||_1}$, we deduce $|||C^n|||_1 = o(d^n)$ and the first item of the lemma follows.

Suppose now that $\rho_r^{\text{smooth}} < d$. Then

$$\rho(A_r^{\text{smooth}}) = \lim_{n \rightarrow \infty} \sqrt[n]{||| (A_r^{\text{smooth}})^n |||_1} < d$$

implies the second item. \square

Proof of theorem 24 — We can suppose that there exists some $i \geq 1$ such that $\beta_i \neq 0$. By hypothesis (ii), there exists a finite set $\Sigma_0 \subset X(\overline{\mathbb{F}}_q)$, such that \mathcal{G}_{Σ_0} is d -regular, strongly connected. Let $r_0 \geq 1$ be the smallest integer such that $\Sigma_0 \subset X(\mathbb{F}_{q^{r_0}})$. Then the spectral radius $\rho(\mathcal{G}_{r_0}^{\text{smooth}}) = d$, so that, thanks to the first item of lemma 26 together with (i)

$$\begin{aligned} \# \tilde{C}_n(\mathbb{F}_{q^{r_0}}) &= ||| (A_r^{\text{smooth}})^n |||_1 + \text{contribution of desingularization} \\ &= \text{cst} \cdot d^n + o(d^n) + \text{contribution of desingularization} \\ &\sim C_{r_0} d^n + o(d^n) \end{aligned}$$

for some constant $C_{r_0} > 0$. Now, Weil bound applied to $\tilde{C}_n(\mathbb{F}_{q^{r_0}})$ for each n implies that $\frac{\# \tilde{C}_n(\mathbb{F}_{q^{r_0}})}{g_n}$ does not tend to infinity, hence $\ell = \liminf \frac{g_n}{d^n} > 0$. Moreover, if $\limsup \frac{g_n}{d^n}$ would be $+\infty$, then we would have $\beta_r = 0$ for all $r \geq 1$ since $\# \tilde{C}_n(\mathbb{F}_{q^r}) \leq X(\mathbb{F}_{q^r}) \times d^n$, which is not the case by assumption. Hence, up to extracting a subsequence, we can assume that

$$\ell = \lim \frac{g_n}{d^n} \in]0, +\infty[.$$

First, we deduce that there exists $r|r_0$, such that $\beta_r > 0$. Indeed,

$$\sum_{r|r_0} r \beta_r \sim \frac{\# \tilde{C}_n(\mathbb{F}_{q^{r_0}})}{g_n} \sim \frac{C_{r_0} d^n}{\ell d^n} = \frac{C_{r_0}}{\ell} > 0.$$

Second, we prove that $\beta_r = 0$ for any $r|r_0$, $r \neq r_0$ (hence $\beta_{r_0} \neq 0$ by the first step above). Suppose by contradiction that there do exist some $r_1|r_0$, $r_1 \neq r_0$, and $\beta_{r_1} > 0$. Then

$$\frac{\# \tilde{C}_n(\mathbb{F}_{q^{r_1}})}{g_n} \geq r \beta_{r_1} > 0.$$

But by (i), the contribution of the normalization is negligible, hence $||| (A_r^{smooth})^n |||_1 \sim C_{r_1} \times d^n$ for some $C_{r_1} > 0$, so that $\rho(\mathcal{G}_{r_1}^{smooth}) = d$ by lemma 26. Lemma 25 applied to $S_{r_1}^{smooth}$ would then imply the existence of some d -regular strongly connected component $\mathcal{G}_{\Sigma_{r_1}}$ of \mathcal{G}_{r_1} , which contradicts the minimality of r_0 for this property !

Third, we prove that for any $r \neq r_0$, we have $\beta_r = 0$. Suppose again by contradiction that there do exists some r_2 not dividing r_0 and $\beta_{r_2} > 0$. Then

$$\frac{\#\tilde{C}_n(\mathbb{F}_{q^{r_2}})}{g_n} \geq r_2 \beta_{r_2} > 0.$$

By (i) and lemma 26 again, we would have $\rho(\mathcal{G}_{r_2}^{smooth}) = d$, and by lemma 25 there would exists some d -regular strongly connected component $\mathcal{G}_{\Sigma_{r_2}}$ of \mathcal{G}_{r_2} . Since r_2 doesn't divide r_0 , we have $\Sigma_{r_1} \neq \Sigma_0$, which contradicts theorem 23. \square

Corollary 27. *Consider Bezerra-Garcia-Stichtenoth's tower defined in section 3.3 over \mathbb{F}_q . Then:*

(i) *one has*

$$\beta_3 = \frac{2(q^2 - 1)}{3(q + 2)}, \quad \beta_r = 0, r \neq 3, \quad \lambda_3 = \frac{2(q^2 - 1)}{(q + 2)};$$

(ii) *the defect δ_{BGS} of this tower, as defined in [TV02], is given by*

$$\delta_{BGS} = 1 - \frac{2(q^2 - 1)}{(q + 2)\sqrt{q^3 - 1}};$$

(iii) *the zeta function of this tower, as defined in [TV02], is given by*

$$Z_{BGS}(T) = \frac{1}{(1 - T)^{\beta_3}}.$$

Remarks

1. Assertion (ii) states that for q large, we have $\delta_{BGS} = 1 - \frac{2}{\sqrt{q}} + o\left(\frac{1}{\sqrt{q}}\right)$, so that the tower is good, but far from being optimal in the sense of [TV02] (see section 1.3).

2. Assertion (iii) indicates that the definition zeta function of a tower, as defined in [TV02], should not be the right one. Note that this is not a rational function since $\frac{2(q^2-1)}{3(q+2)} \notin \mathbb{Z}$.

Proof — This tower is known to be irreducible from [BGS08]. Proposition 3.1 in [BGS05] states that some explicit set $\Omega \subset \mathbb{P}^1(\mathbb{F}_{q^3})$ of cardinality $q(q + 1)$ is forward complete and outside the singular graph \mathcal{G}_{sing} . Since it is finite, it is complete by lemma 13, hence d -regular and strongly connected, so that assumption (H2) of theorem 24 holds true. By the second item of theorem 15, assumption (H1) also holds true since $d = q$, hence there exists at most one r such that β_r is non-zero. Since $\Omega \subset \mathbb{P}^1(\mathbb{F}_{q^3})$ but $\Omega \not\subset \mathbb{P}^1(\mathbb{F}_q)$, we have $\beta_3 \neq 0$. Now, the number of rational points of \tilde{C}_n over \mathbb{F}_{q^3} equals the number of smooth \mathbb{F}_{q^3} -rational points of C_n plus the number of \mathbb{F}_{q^3} -rational points coming from the desingularization of C_n . The number of smooth points in $C_n(\mathbb{F}_{q^3})$ is the sum of the entries of $(A_3^{smooth})^n$, where A_3^{smooth} is the adjacency matrix of $\mathcal{G}_3 \setminus \mathcal{G}_{sing}$. This graph contains only \mathcal{G}_Ω as finite strongly connected subgraph by theorem 23, hence by lemma 25 below $d = q$ is a simple eigenvalue of A_3^{smooth} . From lemma 26 we deduce that the number of smooth points in $C_n(\mathbb{F}_{q^3})$ equals $\#S \times q^n + o(q^n)$. Since the number of geometric points of \tilde{C}_n coming from desingularization is negligible in front of q^n from the last item of theorem 15 again, we have

$$\#\tilde{C}_n(\mathbb{F}_{q^3}) = \#\Omega \times q^n + o(q^n).$$

Finally, the genus of \widetilde{C}_n is given in the first item of theorem 15, so that the first item of the corollary follows. Items (ii) and (iii) follows immediately by definitions of δ_{BGS} and $Z_{BGS}(T)$. \square

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